

EFFECTS OF LARGE DEGENERATE ADVECTION AND BOUNDARY CONDITIONS ON THE PRINCIPAL EIGENVALUE AND ITS EIGENFUNCTION OF A LINEAR SECOND ORDER ELLIPTIC OPERATOR

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Abstract: In this article, we study, as the coefficient $s \rightarrow \infty$, the asymptotic behavior of the principal eigenvalue of the eigenvalue problem

$$-\varphi''(x) - 2sm'(x)\varphi'(x) + c(x)\varphi(x) = \lambda(s)\varphi(x), \quad 0 < x < 1,$$

complemented by a general boundary condition. This problem is relevant to nonlinear propagation phenomena in reaction-diffusion equations. The main point is that the advection (or drift) term m allows natural degeneracy. For instance, m can be constant on $[a, b] \subset [0, 1]$. Depending on the behavior of m near the neighbourhood of the endpoints a and b , the limiting value could be the principal eigenvalue of

$$-\varphi''(x) + c(x)\varphi(x) = \lambda\varphi(x), \quad a < x < b,$$

coupled with Dirichlet or Newmann boundary condition at a and b . A complete understanding of the limiting behavior of the principal eigenvalue and its eigenfunction is obtained, and new fundamental effects of large degenerate advection and boundary conditions on the principal eigenvalue and the principal eigenfunction are revealed. In one space dimension, the results in the existing literature are substantially improved.

Key words and phrases: Principal eigenvalue; Principal eigenfunction; Elliptic operator; Large degenerate advection; Boundary Condition; Asymptotic behavior.

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1. INTRODUCTION

In this article, we are concerned with the linear second order elliptic eigenvalue problem with a general boundary condition in one space dimension:

$$(1.1) \quad \begin{cases} -\varphi''(x) - 2sm'(x)\varphi'(x) + c(x)\varphi(x) = \lambda\varphi(x), & 0 < x < 1, \\ -\hbar_1\varphi'(0) + \ell_1\varphi(0) = \hbar_2\varphi'(1) + \ell_2\varphi(1) = 0, \end{cases}$$

where $m \in C^2([0, 1])$, $c \in C([0, 1])$, and $s \in \mathbb{R}$ is a parameter appearing in front of the advection (or drift) term m , and the nonnegative constants \hbar_i, ℓ_i ($i = 1, 2$) satisfy $\hbar_i + \ell_i > 0$ ($i = 1, 2$).

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It is well known that, given m , c and s , problem (1.1) admits a principal eigenvalue $\lambda = \lambda_1(s) \in \mathbb{R}$, which is unique in the sense that only such an eigenvalue corresponds to a positive eigenfunction φ (φ is also unique up to multiplication). Such a function φ is usually called a principal eigenfunction.

As pointed out by Berestycki, Hamel and Nadirashvili in their remarkable work [2], the qualitative behavior of the eigenvalue problem (1.1) usually plays a significant role in the study of nonlinear propagation phenomena of reaction-diffusion equations.

Therefore, the main goal of this paper is to determine, for a general advection function m , as $s \rightarrow \infty$, the limiting behaviors of the principal eigenvalue and its eigenfunction to (1.1). Problem (1.1) with different types of degeneracies will be treated in a uniform manner. The obtained results will clearly demonstrate that how the interplay between the degeneracy of the advection function and the boundary conditions affects the qualitative properties of the principal eigenvalue and its principal eigenfunction in a substantial way. As far as we know, the current work seems to be the first to reveal such interesting and fundamental influences. Besides, we believe that our results will have natural applications to reaction-diffusion equations.

In the rest of the introduction, we first briefly recall the existing works on (1.1) in the literature and then state our main findings.

1.1. Existing studies. Consider the eigenvalue problem with Neumann boundary condition (i.e., $\ell_1 = \ell_2 = 0$ in (1.1)):

$$(1.2) \quad -\varphi''(x) - 2sm'(x)\varphi'(x) + c(x)\varphi(x) = \lambda\varphi(x), \quad 0 < x < 1; \quad \varphi'(0) = \varphi'(1) = 0,$$

and denote $\lambda_1^{\mathcal{N}}(s)$ and φ to be its principal eigenvalue and eigenfunction, respectively.

It is known that $\lambda_1^{\mathcal{N}}(s)$ enjoys the following variational characterization:

$$(1.3) \quad \lambda_1^{\mathcal{N}}(s) = \min_{\int_0^1 e^{2sm} \varphi^2 dx = 1} \int_0^1 e^{2sm} [(\varphi')^2 + c\varphi^2] dx.$$

Using the substitution $w = e^{sm}\varphi$ in (1.2), one easily sees that $(\lambda_1^{\mathcal{N}}(s), w)$ satisfies

$$(1.4) \quad \begin{cases} -w''(x) + [s^2(m'(x))^2 + sm''(x) + c(x)]w(x) = \lambda_1^{\mathcal{N}}(s)w(x), & 0 < x < 1, \\ w'(0) - sw(0)m'(0) = w'(1) - sw(1)m'(1) = 0. \end{cases}$$

In light of (1.3), the principal eigenvalue $\lambda_1^{\mathcal{N}}(s)$ can be equivalently characterized by

$$(1.5) \quad \lambda_1^{\mathcal{N}}(s) = \min_{\int_0^1 w^2 dx = 1} \int_0^1 [(w' - swm')^2 + cw^2] dx.$$

For later purpose, denote by $w(s, \cdot)$ the positive solution of (1.4) corresponding to $\lambda_1^{\mathcal{N}}(s)$, and normalize it by $\int_0^1 w^2(s, x) dx = 1$ for each $s > 0$. In addition, it is immediately observed that

$$\min_{x \in [0,1]} c(x) \leq \lambda_1^{\mathcal{N}}(s) \leq \max_{x \in [0,1]} c(x), \quad \forall s \in \mathbb{R}.$$

In [3], Chen and Lou investigated the asymptotic behavior of $\lambda_1^{\mathcal{N}}(s)$ as $s \rightarrow \infty$. To present one of their main results, we need recall some definitions introduced there. In the one-dimensional setting, Chen and Lou in [3] said that

- An *interior critical point* of the function m is a point $x \in (0, 1)$ satisfying $m'(x) = 0$, and an interior critical point x is called *non-degenerate* if $m''(x) \neq 0$.
- The boundary points 0 and 1 are always called critical, and a boundary critical point $x \in \{0, 1\}$ is called *non-degenerate* if either $m'(x) \neq 0$ or $m''(x) \neq 0$.
- A *point of local maximum* of m is a point $x \in [0, 1]$ that satisfies $m(x) \geq m(y)$ for every y in a small neighborhood of x , and there exists some sequence $\{r_j\}_{j=1}^{\infty}$ of positive numbers such that

$$m(x) > \max_{[0,1] \cap \{x-r_j, x+r_j\}} m, \quad \forall j \geq 1, \quad \lim_{j \rightarrow \infty} r_j = 0.$$

As remarked by [3], the reason the authors used such a definition of local maximum is that they had to avoid the occurrence of the situation that the set of local maximum of m contains some flat piece. Then one main result—Theorem 1 of [3] concludes that

Theorem 1.1. *Assume that all critical points of m are non-degenerate. Let \mathcal{M} be the set of points of local maximum of m . Then $\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{N}}(s) = \min_{x \in \mathcal{M}} c(x)$.*

Theorem 1.1 is of significant importance, and it has found new interesting applications in several classical reaction-diffusion-advection problems arising from ecology; for example, [3, 10, 11, 12] to list a few. Obviously, Theorem 1.1 deals with only the case that m has finitely many non-degenerate isolated points of local maximum. Later on, in the companion paper [4], Chen and Lou studied the limiting behavior of $\lambda_1^{\mathcal{N}}(s)$ when the diffusion and advection rates are both large or small.

In [2], Berestycki, Hamel and Nadinashvili investigated problems like (1.1) with $c(x) \equiv 0$ and under either Dirichlet, Neumann or periodic boundary condition in arbitrary space dimensions. They focussed on the situation that the drift velocity (advection term) \mathbf{v} is divergence-free, and established the equivalent connections between the boundedness of the principal eigenvalue with regard to large drift and the existence of the first integrals of the velocity field \mathbf{v} . As consequences of their results, important influences of large advection or drift on the speed of propagation of pulsating travelling fronts were revealed there.

As far as the periodic boundary problem is concerned, we assume that $m \in C^2(\mathbb{R})$ and $c \in C(\mathbb{R})$ and both of them are 1-periodic (that is, $f(x) = f(x+1)$, $f \in \{m, c\}$, $\forall x \in \mathbb{R}$). Then, there is a unique principal eigenvalue to the eigenvalue problem:

$$(1.6) \quad -\varphi''(x) - 2sm'(x)\varphi'(x) + c(x)\varphi(x) = \lambda\varphi(x), \quad x \in \mathbb{R}; \quad \varphi(x) = \varphi(x+1), \quad x \in \mathbb{R}.$$

If one attempts to apply the result of [2] to study the limiting behavior of the principal eigenvalue of problems (1.2) and (1.6) as $s \rightarrow \infty$, the restricted condition on m imposed there now reduces to require that m is constant on $[0, 1]$. This is a trivial case. Here, of our interest is a general nonconstant function m so that spatial heterogeneity of environment can be reflected.

When $\hbar_1 = \hbar_2 = 0$ in (1.1), we have the Dirichlet eigenvalue problem:

$$(1.7) \quad -\varphi''(x) - 2sm'(x)\varphi'(x) + c(x)\varphi(x) = \lambda\varphi(x), \quad 0 < x < 1; \quad \varphi(0) = \varphi(1) = 0.$$

Then, Theorem 0.3 of [2], one of the main results, only covers the case of $m(x) = ax$ on $[0, 1]$ for some constant a , and states that the principal eigenvalue of (1.7) is bounded as $s \rightarrow \infty$ if

and only if $a = 0$. It turns out that this is a very special case to be treated in our present work. Indeed, as long as m' changes sign at most finitely many times, as $s \rightarrow \infty$, we are able to derive a necessary and sufficient condition to guarantee the boundedness of the principal eigenvalue of the general problem (1.1). Moreover, once the principal eigenvalue is bounded with regard to large s , the asymptotic behaviors of the principal eigenvalue and its eigenfunction will be precisely given. See our main results: Theorems 1.2, 1.3 and 1.4 below. We would like to point out that the analysis of [2] seems inapplicable to the general problem (1.1).

In the direction of research on the effect of large advection on the principal eigenvalue, it is worth mentioning a series of impressive work [5, 6, 8], done by Friedman and his coauthors more than 40 years ago, which concerned the Dirichlet boundary condition case and obtained refined upper and lower bounds for the principal eigenvalue when the advection coefficient is large. Nevertheless, for (1.7), their results seem to apply only to a few special kinds of m ; for example, $m'(x)$ changes sign on $[0, 1]$ at most once. Regarding other related works, one may refer to [7, 13] and the references therein. We further remark that no information of the associated principal eigenfunction was provided in [2, 5, 6, 7, 8, 13].

1.2. Our main results. As mentioned before, the objective of the present paper is to determine, for a general advection function m , as $s \rightarrow \infty$, the limiting behaviors of the principal eigenvalue and its eigenfunction to problems (1.1) and (1.6). Throughout the paper, unless otherwise specified, we always assume that

$$(1.8) \quad m \text{ is not constant and } m'(x) \text{ changes sign at most finitely many times on } [0, 1].$$

Hence, here we allow m to have various natural kinds of degeneracy.

Before stating the main results of this paper, we need to classify the set of points of local maximum points of m and then introduce necessary notation. We call that

- A *point of local maximum* of m is a point $x \in [0, 1]$ such that there is a small $\epsilon_0 > 0$ such that $m(x) \geq m(y)$ in $(x - \epsilon_0, x + \epsilon_0) \cap [0, 1]$, and such x is said to be an *interior point of local maximum* if $x \in (0, 1)$;
- An *isolated point of local maximum* of m is a point $x^I \in [0, 1]$ such that there is a small $\epsilon_0 > 0$ such that $m(x^I) > m(x)$ in $(x^I - \epsilon_0, x^I + \epsilon_0) \cap [0, 1] \setminus \{x^I\}$;
- A *segment of local maximum* of m is a closed interval $[a, b] \subset [0, 1]$ such that there is a small $\epsilon_0 > 0$ such that m is constant on the closed interval $[a, b]$, m is monotone (non-increasing or non-decreasing) on $[a - \epsilon_0, a] \cap [0, 1]$ and $[b, b + \epsilon_0] \cap [0, 1]$, and for any small $\epsilon > 0$, $\{x \in [0, 1] : m'(x) \neq 0\} \cap (a - \epsilon, a) \neq \emptyset$ if $0 < a$ and $\{x \in [0, 1] : m'(x) \neq 0\} \cap (b, b + \epsilon) \neq \emptyset$ if $b < 1$.

In addition, when m is a 1-periodic function, we make the convention from now on that the isolated point and segment of local maximum of m is understood to be restricted to the one period interval $[0, 1]$ with the same definitions as above.

Under the assumption (1.8), it is clear that m admits at most finitely many isolated points of local maximum. For later purpose, we will have to use different notation to distinguish all possible *segments of local maximum* of m as follows.

$[a^i, b^j]$ with $0 < a^i < b^j < 1$ and $i, j \in \{I, D\}$: m is constant on $[a^i, b^j]$, m is non-decreasing (non-increasing, respectively) in $[a^i - \epsilon_0, a^i]$ for some small $\epsilon_0 > 0$ and $\{x \in [0, 1] : m'(x) > 0\} \cap (a^i - \epsilon, a^i) \neq \emptyset$ ($\{x \in [0, 1] : m'(x) < 0\} \cap (a^i - \epsilon, a^i) \neq \emptyset$, respectively) for any small $\epsilon > 0$ if $i = I$ (if $i = D$, respectively), and m is non-decreasing (non-increasing, respectively) in $[b^j, b^j + \epsilon_0]$ for some small $\epsilon_0 > 0$ and $\{x \in [0, 1] : m'(x) > 0\} \cap (b^j, b^j + \epsilon) \neq \emptyset$ ($\{x \in [0, 1] : m'(x) < 0\} \cap (b^j, b^j + \epsilon) \neq \emptyset$, respectively) for any small $\epsilon > 0$ if $j = I$ (if $j = D$, respectively).

$[0, a^I]$ with $0 < a^I < 1$: m is constant on $[0, a^I]$ and m is non-decreasing in $[a^I, a^I + \epsilon_0]$ for some small $\epsilon_0 > 0$ and $\{x \in [0, 1] : m'(x) > 0\} \cap (a^I, a^I + \epsilon) \neq \emptyset$ for any small $\epsilon > 0$.

$[0, a^D]$ with $0 < a^D < 1$: m is constant on $[0, a^D]$ and m is non-increasing in $[a^D, a^D + \epsilon_0]$ for some small $\epsilon_0 > 0$ and $\{x \in [0, 1] : m'(x) < 0\} \cap (a^D, a^D + \epsilon) \neq \emptyset$ for any small $\epsilon > 0$.

$[a^I, 1]$ with $0 < a^I < 1$: m is constant on $[a^I, 1]$ and m is non-decreasing in $[a^I - \epsilon_0, a^I]$ for some small $\epsilon_0 > 0$ and $\{x \in [0, 1] : m'(x) > 0\} \cap (a^I - \epsilon, a^I) \neq \emptyset$ for any small $\epsilon > 0$.

$[a^D, 1]$ with $0 < a^D < 1$: m is constant on $[a^D, 1]$ and m is non-increasing in $[a^D - \epsilon_0, a^D]$ for some small $\epsilon_0 > 0$ and $\{x \in [0, 1] : m'(x) < 0\} \cap (a^D - \epsilon, a^D) \neq \emptyset$ for any small $\epsilon > 0$.

In the above, the capital letters I and D represent increasing (i.e., non-decreasing) and decreasing (i.e., non-increasing), respectively; for instance, $[a^I, b^D]$ means that m increases in a left neighbourhood of a^I and decreases in a right neighbourhood of b^D while m is constant on $[a^I, b^D]$.

Under the assumption (1.8), it is noted that m admits at most finitely many segments $[a^I, b^D]$ and $[a^D, b^I]$ of local maximum, and it has at most countably many segments $[a^I, b^I]$ and $[a^D, b^D]$ of local maximum. We need more notation as follows.

Given a closed interval $[a, b] \subset [0, 1]$ with $0 < a < b < 1$ and $i, j \in \{\mathcal{N}, \mathcal{D}\}$, we denote by $\lambda_1^{ij}(a, b)$ the principal eigenvalue of the elliptic eigenvalue problem:

$$-\varphi''(x) + c(x)\varphi(x) = \lambda\varphi(x), \quad a < x < b$$

subject to Neumann boundary condition (Dirichlet boundary condition, respectively) at the left boundary point a if $i = \mathcal{N}$ (if $i = \mathcal{D}$, respectively), and Neumann boundary condition (Dirichlet boundary condition, respectively) at the right boundary point b if $j = \mathcal{N}$ (if $j = \mathcal{D}$, respectively).

Given $[0, b] \subset [0, 1]$ with $0 < b < 1$ and $i \in \{\mathcal{N}, \mathcal{D}\}$, denote by $\lambda_1^{\mathcal{R}i}(0, b)$ the principal eigenvalue of

$$-\varphi''(x) + c(x)\varphi(x) = \lambda\varphi(x), \quad 0 < x < b; \quad -\hbar_1\varphi'(0) + \ell_1\varphi(0) = 0$$

with Neumann boundary condition (Dirichlet boundary condition, respectively) at b if $i = \mathcal{N}$ (if $i = \mathcal{D}$, respectively).

Given $[a, 1] \subset [0, 1]$ with $0 < a < 1$ and $i \in \{\mathcal{N}, \mathcal{D}\}$, denote by $\lambda_1^{\mathcal{R}i}(a, 1)$ the principal eigenvalue of

$$-\varphi''(x) + c(x)\varphi(x) = \lambda\varphi(x), \quad a < x < 1; \quad \hbar_2\varphi'(1) + \ell_2\varphi(1) = 0$$

with Neumann boundary condition (Dirichlet boundary condition, respectively) at a if $i = \mathcal{N}$ (if $i = \mathcal{D}$, respectively).

According to the assumption (1.8), it is obviously seen that the set of points of local maximum of m can be represented by

$$(1.9) \quad \mathcal{M} = \cup_{i=1}^9 \mathcal{M}_i,$$

in which

$$\begin{aligned} \mathcal{M}_1 &= \{x_i^I\}_{i=1}^{h_1}, \quad \mathcal{M}_2 = \{[a_i^I, b_i^I]\}_{i=1}^{h_2}, \quad \mathcal{M}_3 = \{[a_i^I, b_i^D]\}_{i=1}^{h_3}, \quad \mathcal{M}_4 = \{[a_i^D, b_i^I]\}_{i=1}^{h_4}, \\ \mathcal{M}_5 &= \{[a_i^D, b_i^D]\}_{i=1}^{h_5}, \quad \mathcal{M}_6 \subset \{[0, a^I]\}, \quad \mathcal{M}_7 \subset \{[0, a^D]\}, \quad \mathcal{M}_8 \subset \{[a^I, 1]\}, \quad \mathcal{M}_9 \subset \{[a^D, 1]\}, \end{aligned}$$

where h_1, h_3, h_4 are finite integers while h_2, h_5 may be finite integers or infinity. We allow some \mathcal{M}_i to be empty. According to our definitions, $x \cap y = \emptyset$ for any $x, y \in \mathcal{M}_i$, $x \neq y$, $i \in \{2, 3, 4, 5\}$, and $x \cap y = \emptyset$ for any $x \in \mathcal{M}_i$, $y \in \mathcal{M}_j$ if $i \neq j$. Moreover, either \mathcal{M}_6 or \mathcal{M}_7 or both of them must be an empty set, and the same is true for \mathcal{M}_8 and \mathcal{M}_9 . Though some \mathcal{M}_i may be empty, it is apparent that $\mathcal{M} \neq \emptyset$.

For simplicity, let us also set

$$\begin{aligned} \mathfrak{L} = \min \bigg\{ & \min\{\lambda_1^{\mathcal{ND}}(a_i^I, b_i^I) : [a_i^I, b_i^I] \in \mathcal{M}_2\}, \quad \min\{\lambda_1^{\mathcal{NN}}(a_i^I, b_i^D) : [a_i^I, b_i^D] \in \mathcal{M}_3\}, \\ & \min\{\lambda_1^{\mathcal{DD}}(a_i^D, b_i^I) : [a_i^D, b_i^I] \in \mathcal{M}_4\}, \quad \min\{\lambda_1^{\mathcal{DN}}(a_i^D, b_i^D) : [a_i^D, b_i^D] \in \mathcal{M}_5\} \bigg\}. \end{aligned}$$

We want to stress that $\min\{\lambda_1^{\mathcal{ND}}(a_i^I, b_i^I) : [a_i^I, b_i^I] \in \mathcal{M}_2\}$ and $\min\{\lambda_1^{\mathcal{DN}}(a_i^D, b_i^D) : [a_i^D, b_i^D] \in \mathcal{M}_5\}$ are well defined since $\lambda_1^{\mathcal{ND}}(a_i^I, b_i^I) \rightarrow \infty$ when $b_i^I - a_i^I \rightarrow 0$ and $\lambda_1^{\mathcal{DN}}(a_i^D, b_i^D) \rightarrow \infty$ when $b_i^D - a_i^D \rightarrow 0$.

Our first result concerns the limiting behavior of the principal eigenvalue $\lambda_1(s)$ for problem (1.1), and reads as follows.

Theorem 1.2. *Assume that the set \mathcal{M} of points of local maximum of m is given by (1.9). Then the following assertions hold.*

- (i) $\lim_{s \rightarrow \infty} \lambda_1(s) = \infty$ if and only if
 - (i-1) $\mathcal{M} = \mathcal{M}_1 \subset \{0, 1\}$ when $\ell_1 > 0$ and $\ell_2 > 0$;
 - (i-2) $\mathcal{M} = \mathcal{M}_1 = \{0\}$ when $\ell_1 > 0$ and $\ell_2 = 0$;
 - (i-3) $\mathcal{M} = \mathcal{M}_1 = \{1\}$ when $\ell_1 = 0$ and $\ell_2 > 0$.
- (ii) If $\lim_{s \rightarrow \infty} \lambda_1(s) < \infty$, then

$$\begin{aligned} \lim_{s \rightarrow \infty} \lambda_1(s) = \min \bigg\{ & \mathfrak{L}, \quad \min\{c(x) : x \in \mathcal{M}_1 \setminus \{0, 1\}\}, \quad \lambda_1^{\mathcal{RD}}(0, a^I) \text{ (if } [0, a^I] \in \mathcal{M}_6\text{)}, \\ & \lambda_1^{\mathcal{RN}}(0, a^D) \text{ (if } [0, a^D] \in \mathcal{M}_7\text{)}, \quad \lambda_1^{\mathcal{NR}}(a^I, 1) \text{ (if } [a^I, 1] \in \mathcal{M}_8\text{)}, \\ & \lambda_1^{\mathcal{DR}}(a^D, 1) \text{ (if } [a^D, 1] \in \mathcal{M}_9\text{)}, \quad c(0) \text{ (if } 0 \in \mathcal{M}_1 \text{ and } \ell_1 = 0\text{)}, \\ & c(1) \text{ (if } 1 \in \mathcal{M}_1 \text{ and } \ell_2 = 0\text{)} \bigg\}. \end{aligned}$$

Remark 1.1. *Concerning Theorem 1.2, we would like to make the following comments.*

- (i) *In comparison with Theorem 1.1, when the Neumann problem (1.2) is concerned, Theorem 1.2 covers the case that m has segments of local maximum; and even if m has isolated points of local maximum, unlike Theorem 1.1, we do not impose non-degeneracy condition on those points.*

- (ii) From Theorem 1.2, we can see that when 0 or 1 is an isolated point of local maximum of m , the limiting value of the principal eigenvalue, when finite, is affected by such a boundary point only if the Neumann boundary condition is prescribed there.
- (iii) When $\ell_1 + \ell_2 > 0$, in view of Theorem 1.2, we have the following observations.
 - (iii-1) If $\ell_1, \ell_2 > 0$, then $\lim_{s \rightarrow \infty} \lambda_1(s) = \infty$ if and only if either m is strictly increasing or strictly decreasing on $[0, 1]$, or m is strictly decreasing on $[0, x_0]$ while strictly increasing on $[x_0, 1]$ for some $x_0 \in (0, 1)$.
 - (iii-2) If $\ell_1 = 0, \ell_2 > 0$, then $\lim_{s \rightarrow \infty} \lambda_1(s) = \infty$ if and only if m is strictly increasing $[0, 1]$.
 - (iii-3) If $\ell_1 > 0, \ell_2 = 0$, then $\lim_{s \rightarrow \infty} \lambda_1(s) = \infty$ if and only if m is strictly decreasing $[0, 1]$.
- (iv) For the Dirichlet eigenvalue problem (1.7), the main results of [5] tell us that if $m'(x) > 0$ or $m'(x) < 0$ on $[0, 1]$, then there exists a constant $\vartheta > 1$ such that

$$s^2/\vartheta \leq \lambda_1(s) \leq \vartheta s^2 \quad \text{as } s \rightarrow \infty,$$

and if $m'(x) < 0$ in $[0, x_0)$, $m'(x) > 0$ in $(x_0, 1]$, and $|x - x_0|^{1+\nu}/\sigma \leq |m'(x)| \leq \sigma|x - x_0|^{1+\nu}$, $\forall x \in [0, 1]$ for some $x_0 \in (0, 1)$ and constants $\sigma > 1, \nu > 0$, then there exists a constant $\vartheta > 1$ such that

$$s^{\frac{2}{\nu+1}}/\vartheta \leq \lambda_1(s) \leq \vartheta s^{\frac{2}{\nu+1}} \quad \text{as } s \rightarrow \infty.$$

On the other hand, [8] showed that if $c(x) = c$ is constant, $m'(0) > 0$, $m'(1) < 0$, and m has only one isolated interior point of local maximum in $(0, 1)$, then $\lambda_1(s) = c + O(se^{-s})$ as $s \rightarrow \infty$. However, the asymptotic growth rate of $\lambda_1(s)$ is not known in general.

We next consider the periodic eigenvalue problem (1.6). Without loss of generality, we assume that $m'(0) > 0$ (and so $m'(1) > 0$) due to 1-periodicity of m . Then the following result holds.

Theorem 1.3. Assume that $m'(0) > 0$ and the set of points of local maximum of m is given by (1.9), and denote by $\lambda_1^{\mathcal{P}}(s)$ the principal eigenvalue of (1.6). Then

$$\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{P}}(s) = \min \left\{ \mathfrak{L}, \min \{c(x) : x \in \mathcal{M}_1\} \right\}.$$

We now turn our attention to the limiting profile of the principal eigenfunction. In what follows, for sake of simplicity we only state the result for problem (1.2); for the general problem (1.1) and the periodic problem (1.6), the analogous result remains true.

As in [3], we define

$$\lambda^* = \limsup_{s \rightarrow \infty} \lambda_1^{\mathcal{N}}(s), \quad \lambda_* = \liminf_{s \rightarrow \infty} \lambda_1^{\mathcal{N}}(s).$$

Recall that $w(s, \cdot)$ is the positive solution of (1.4) corresponding to the principal eigenvalue $\lambda_1^{\mathcal{N}}(s)$ with the normalization $\int_0^1 w^2(s, x) dx = 1$ for each $s > 0$. It is well known that the sequence $\{w^2(s, \cdot)\}_{s>0}$ is weakly compact in the space $L^1(0, 1)$. This implies that there exists a sequence $\{s_j\}_{j=1}^\infty$ satisfying $s_j \rightarrow \infty$ as $j \rightarrow \infty$, such that

$$(1.10) \quad \lim_{j \rightarrow \infty} \int_0^1 w^2(s_j, x) \zeta(x) dx = \int_{[0,1]} \zeta(x) \mu(dx), \quad \forall \zeta \in C([0, 1])$$

for a certain probability measure μ . Therefore, from (1.5) and (1.10), it follows that

$$(1.11) \quad \lambda_* \geq \lim_{j \rightarrow \infty} \int_0^1 w^2(s_j, x) c(x) dx = \int_{[0,1]} c(x) \mu(dx).$$

We first see from Lemmas 2.5 and 2.6 and their proofs that, roughly speaking, in the set $E \subset [0, 1]$ where m has no local maximum, $\mu(E) = 0$ and $w(s, \cdot) \rightarrow 0$ in E as $s \rightarrow \infty$. We are now interested in the asymptotic behavior of the principal eigenfunction $w(s, \cdot)$ in an isolated point or segment of local maximum of m carrying a positive Radon measure of μ . More precisely, we have

Theorem 1.4. *Let $w(s, \cdot)$, normalized by $\int_0^1 w^2(s, x) dx = 1$ for each $s > 0$, be the principal eigenfunction of (1.2) corresponding to $\lambda_1^{\mathcal{N}}(s)$, and μ be the Radon measure defined through the sequence $\{s_j\}$ in (1.10). The following assertions hold.*

(1) *Assume that $x_0 \in \mathcal{M}_1$ and satisfies*

$$m^{(k)}(x_0) = 0, \quad \forall 1 \leq k \leq k^* - 1, \quad \text{and} \quad m^{(k^*)}(x_0) \neq 0,$$

for some integer $k^ \geq 2$, and $\mu(\{x_0\}) > 0$. Then $\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{N}}(s) = c(x_0)$. Moreover, up to a subsequence of $\{s_j\}$, we have*

(1-i) *If $x_0 \in (0, 1)$, then*

$$\mu(\{x_0\})^{-\frac{1}{2}} s^{\frac{1}{2k^*}} w(s, x_0 + s^{-\frac{1}{k^*}} y) \rightarrow W^* \text{ locally uniformly in } \mathbb{R},$$

where W^ with $\int_{-\infty}^{\infty} (W^*)^2(y) dy = 1$ is a positive solution of the linear ODE equation*

$$(W^*)''(y) = ((m^{(k^*)}(x_0))^2 y^{2(k^*-1)} + m^{(k^*)}(x_0) y^{k^*-2}) W^*(y), \quad y \in \mathbb{R}.$$

(1-ii) *If $x_0 = 0$ or $x_0 = 1$, then*

$$\mu(\{x_0\})^{-\frac{1}{2}} s^{\frac{1}{2k^*}} w(s, x_0 + s^{-\frac{1}{k^*}} y) \rightarrow W_* \text{ locally uniformly in } \mathbb{R}_*,$$

where $\mathbb{R}_ = (0, \infty)$ if $x_0 = 0$ and $\mathbb{R}_* = (-\infty, 0)$ if $x_0 = 1$, and W_* with $\int_{\mathbb{R}_*} (W_*)^2(y) dy = 1$ is a positive solution of the linear ODE equation*

$$(W_*)''(y) = ((m^{(k^*)}(x_0))^2 y^{2(k^*-1)} + m^{(k^*)}(x_0) y^{k^*-2}) W_*(y), \quad y \in \mathbb{R}_*.$$

(2) *Assume that $[a, b] \in \mathcal{M}_2$ and $\mu((a, b)) > 0$. Then $\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{N}}(s) = \lambda_1^{\mathcal{N}\mathcal{D}}(a, b)$, and $w(s, \cdot) \rightarrow \varphi_1$ in $C^1([a, b])$, where φ_1 is an eigenfunction corresponding to $\lambda_1^{\mathcal{N}\mathcal{D}}(a, b)$.*

(3) *Assume that $[a, b] \in \mathcal{M}_3$ and $\mu((a, b)) > 0$. Then $\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{N}}(s) = \lambda_1^{\mathcal{N}\mathcal{N}}(a, b)$, and $w(s, \cdot) \rightarrow \varphi_1$ in $C^1([a, b])$, where φ_1 is an eigenfunction corresponding to $\lambda_1^{\mathcal{N}\mathcal{N}}(a, b)$.*

(4) *Assume that $[a, b] \in \mathcal{M}_4$ and $\mu((a, b)) > 0$. Then $\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{N}}(s) = \lambda_1^{\mathcal{D}\mathcal{D}}(a, b)$, and $w(s, \cdot) \rightarrow \varphi_1$ in $C^1([a, b])$, where φ_1 is an eigenfunction corresponding to $\lambda_1^{\mathcal{D}\mathcal{D}}(a, b)$.*

(5) *Assume that $[a, b] \in \mathcal{M}_5$ and $\mu((a, b)) > 0$. Then $\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{N}}(s) = \lambda_1^{\mathcal{D}\mathcal{N}}(a, b)$, and $w(s, \cdot) \rightarrow \varphi_1$ in $C^1([a, b])$, where φ_1 is an eigenfunction corresponding to $\lambda_1^{\mathcal{D}\mathcal{N}}(a, b)$.*

(6) *Assume that $[0, a^I] \in \mathcal{M}_6$ and $\mu([0, a^I]) > 0$. Then $\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{N}}(s) = \lambda_1^{\mathcal{N}\mathcal{D}}(0, a^I)$, and $w(s, \cdot) \rightarrow \varphi_1$ in $C^1([0, a^I])$, where φ_1 is an eigenfunction corresponding to $\lambda_1^{\mathcal{N}\mathcal{D}}(0, a^I)$.*

(7) *Assume that $[0, a^D] \in \mathcal{M}_7$ and $\mu([0, a^D]) > 0$. Then $\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{N}}(s) = \lambda_1^{\mathcal{N}\mathcal{N}}(0, a^D)$, and $w(s, \cdot) \rightarrow \varphi_1$ in $C^1([0, a^D])$, where φ_1 is an eigenfunction corresponding to $\lambda_1^{\mathcal{N}\mathcal{N}}(0, a^D)$.*

- (8) Assume that $[a^I, 1] \in \mathcal{M}_8$ and $\mu([a^I, 1]) > 0$. Then $\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{N}}(s) = \lambda_1^{\mathcal{NN}}(a^I, 1)$, and $w(s, \cdot) \rightarrow \varphi_1$ in $C^1([a^I, 1])$, where φ_1 is an eigenfunction corresponding to $\lambda_1^{\mathcal{NN}}(a^I, 1)$.
- (9) Assume that $[a^D, 1] \in \mathcal{M}_9$ and $\mu([a^D, 1]) > 0$. Then $\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{N}}(s) = \lambda_1^{\mathcal{DN}}(a^D, 1)$, and $w(s, \cdot) \rightarrow \varphi_1$ in $C^1([a^D, 1])$, where φ_1 is an eigenfunction corresponding to $\lambda_1^{\mathcal{DN}}(a^D, 1)$.

We remark that if $k^* = 2$, Theorem 1.4 is reduced to Theorem 2 of [3] in the one dimension case, and the unique positive solution W^* and W_* can be explicitly given (see Theorem 2 of [3] for the details).

To obtain the results stated above, we mainly follow the approach of [3]. However, in doing so, several new ideas and techniques will have to be introduced in order to overcome a number of highly nontrivial difficulties caused by the degeneracy of m .

Roughly speaking, our strategy consists of two main steps. As a first step, we establish $\limsup_{s \rightarrow \infty} \lambda_1(s)$ by constructing suitable testing functions (due to the variational characterization of $\lambda_1(s)$). In the second step of yielding $\liminf_{s \rightarrow \infty} \lambda_1(s)$, we first need to establish a refined description of the support of the probability measure μ , showing that $\mu([0, 1] \setminus \mathcal{M}) = 0$. Then, combined with the variational characterization of $\lambda_1(s)$ again, among other ingredients, we can derive $\liminf_{s \rightarrow \infty} \lambda_1(s)$, which coincides with $\limsup_{s \rightarrow \infty} \lambda_1(s)$. The asymptotic profile of the eigenfunction $w(s, x)$ is determined by using some local analysis at an isolated point of local maximum of m , and by elliptic regularity theory in a segment of local maximum of m .

In [3], under the assumption of Theorem 1.1, for problem (1.2), Chen and Lou asked if the support of the probability measure μ is precisely given by the set \mathcal{M} of points of local maximum. Theorems 1.2, 1.4 here and their proofs show that this is not the case in general. As a matter of fact, the support of μ consists of the set through which the limit $\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{N}}(s)$ is attained, and conversely, the limit is attained on the support of μ in the same sense as explained in Remark 2.1. Such comments also apply to the general problem (1.1) and the periodic problem (1.6).

To illustrate the main results obtained in this paper, we shall look at the following three typical examples.

Example 1: m is strictly decreasing on $[0, x_1] \cup [x_3, x_4]$, strictly increasing on $[x_1, x_2] \cup [x_4, 1]$ and constant on $[x_2, x_3]$ (See Figure 1). Let x_2, x_3 shrink to one point x_0 so that m is strictly decreasing on $[0, x_1] \cup [x_0, x_4]$ and strictly increasing on $[x_1, x_0] \cup [x_4, 1]$ (See Figure 2).

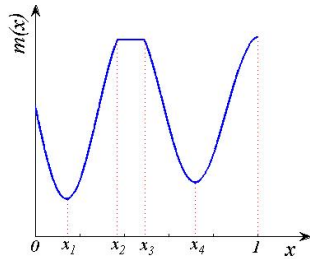


Figure 1

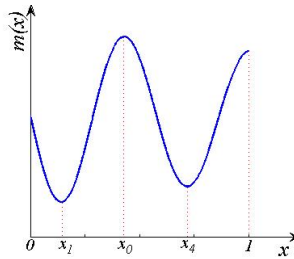


Figure 2

Denote by $\lambda_1^{\mathcal{RR}}(s)$ ($\lambda_1^{\mathcal{NR}}(s)$; $\lambda_1^{\mathcal{RN}}(s)$, respectively) the principal eigenvalue of (1.1) with $\ell_1, \ell_2 > 0$ ($\ell_1 = 0, \ell_2 > 0$; $\ell_1 > 0, \ell_2 = 0$, respectively), and let μ be the probability measure

corresponding to the normalized principal eigenfunction $w(s, \cdot)$ of the associated eigenvalue problem as defined for the Neumann problem (1.2) and (1.4).

Assume that m is given as in Figure 1, we have

- (1) $\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{N}}(s) = \min\{c(0), c(1), \lambda_1^{\mathcal{NN}}(x_2, x_3)\}$, $\mu((0, x_2] \cup [x_3, 1)) = 0$ and $\mu(\{0, 1\} \cup (x_2, x_3)) = 1$.
- (2) $\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{RR}}(s) = \lambda_1^{\mathcal{NN}}(x_2, x_3)$, and $\mu([0, x_2] \cup [x_3, 1]) = 0$ and $\mu((x_2, x_3)) = 1$.
- (3) $\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{NR}}(s) = \min\{c(0), \lambda_1^{\mathcal{NN}}(x_2, x_3)\}$, $\mu((0, x_2] \cup [x_3, 1]) = 0$ and $\mu(\{0\} \cup (x_2, x_3)) = 1$.
- (4) $\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{RN}}(s) = \min\{c(1), \lambda_1^{\mathcal{NN}}(x_2, x_3)\}$, $\mu([0, x_2] \cup [x_3, 1)) = 0$ and $\mu(\{1\} \cup (x_2, x_3)) = 1$.

When x_2 and x_3 shrink to one point x_0 as shown in Figure 2, we have

- (1') $\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{N}}(s) = \min\{c(0), c(1), c(x_0)\}$, $\mu((0, x_0) \cup (x_0, 1)) = 0$ and $\mu(\{0, 1, x_0\}) = 1$.
- (2') $\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{RR}}(s) = c(x_0)$, $\mu([0, x_0] \cup (x_0, 1]) = 0$ and $\mu(\{x_0\}) = 1$.
- (3') $\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{NR}}(s) = \min\{c(0), c(x_0)\}$, $\mu((0, x_0) \cup (x_0, 1]) = 0$ and $\mu(\{0, x_0\}) = 1$.
- (4') $\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{RN}}(s) = \min\{c(1), c(x_0)\}$, $\mu([0, x_0] \cup (x_0, 1)) = 0$ and $\mu(\{x_0, 1\}) = 1$.

We note that $\lambda_1^{\mathcal{NN}}(x_2, x_3)$ converges to $c(x_0)$ as x_2, x_3 shrinks to the point x_0 . Thus, our result coincides with Theorem 1.1 obtained by Chen and Lou [3]; however, we do not require non-degeneracy of m at x_0 .

Example 2: m is strictly decreasing on $[0, x_1]$, strictly increasing on $[x_2, 1]$ and constant on $[x_1, x_2]$ (Figure 3). Let x_1, x_2 shrink to one point x_0 so that m is strictly decreasing on $[0, x_0]$ and strictly increasing on $[x_0, 1]$ (Figure 4). Assume that m is given as in Figure 3, we have

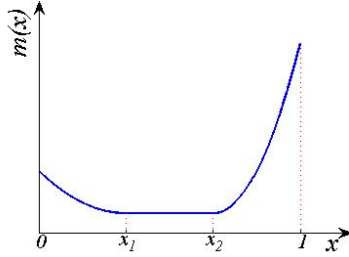


Figure 3

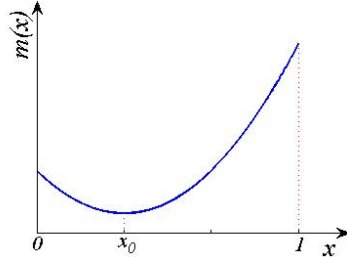


Figure 4

- (1) $\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{N}}(s) = \min\{c(0), c(1), \lambda_1^{\mathcal{DD}}(x_1, x_2)\}$, $\mu((0, x_1] \cup [x_2, 1)) = 0$ and $\mu(\{0, 1\} \cup (x_1, x_2)) = 1$.
- (2) $\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{RR}}(s) = \lambda_1^{\mathcal{DD}}(x_1, x_2)$, $\mu([0, x_1] \cup [x_2, 1]) = 0$ and $\mu((x_1, x_2)) = 1$.
- (3) $\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{NR}}(s) = \min\{c(0), \lambda_1^{\mathcal{DD}}(x_1, x_2)\}$, $\mu((0, x_1] \cup [x_2, 1]) = 0$ and $\mu(\{0\} \cup (x_1, x_2)) = 1$.
- (4) $\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{RN}}(s) = \min\{c(1), \lambda_1^{\mathcal{DD}}(x_1, x_2)\}$, $\mu([0, x_1] \cup [x_2, 1)) = 0$ and $\mu(\{1\} \cup (x_1, x_2)) = 1$.

When x_1 and x_2 shrink to one point x_0 as shown in Figure 4, we have

- (1') $\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{N}}(s) = \min\{c(0), c(1)\}$, $\mu((0, 1)) = 0$ and $\mu(\{0, 1\}) = 1$.
- (2') $\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{RR}}(s) = \infty$.
- (3') $\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{NR}}(s) = c(0)$, $\mu((0, 1]) = 0$ and $\mu(\{0\}) = 1$.
- (4') $\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{RN}}(s) = c(1)$, $\mu([0, 1)) = 0$ and $\mu(\{1\}) = 1$.

Example 3: m is strictly increasing on $[0, x_1] \cup [x_2, 1]$ and constant on $[x_1, x_2]$ (Figure 5). Let x_1, x_2 shrink to one point x_0 so that m is strictly increasing on $[0, 1]$ (Figure 6). Assume that m is given as in Figure 5, we have

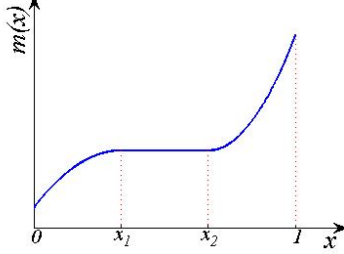


Figure 5

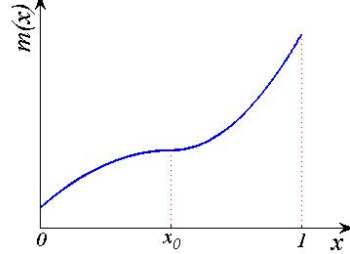


Figure 6

- (1) $\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{N}}(s) = \min\{c(1), \lambda_1^{\mathcal{N}\mathcal{D}}(x_1, x_2)\}$, $\mu([0, x_1] \cup [x_2, 1)) = 0$ and $\mu(\{1\} \cup (x_1, x_2)) = 1$.
- (2) $\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{R}\mathcal{R}}(s) = \lambda_1^{\mathcal{N}\mathcal{D}}(x_1, x_2)$, and $\mu([0, x_1] \cup [x_2, 1]) = 0$ and $\mu((x_1, x_2)) = 1$.
- (3) $\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{N}\mathcal{R}}(s) = \lambda_1^{\mathcal{N}\mathcal{D}}(x_1, x_2)$, $\mu([0, x_1] \cup [x_2, 1]) = 0$ and $\mu((x_1, x_2)) = 1$.
- (4) $\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{R}\mathcal{N}}(s) = \min\{c(1), \lambda_1^{\mathcal{N}\mathcal{D}}(x_1, x_2)\}$, $\mu([0, x_1] \cup [x_2, 1)) = 0$ and $\mu(\{1\} \cup (x_1, x_2)) = 1$.

When x_1 and x_2 shrink to one point x_0 as shown in Figure 6, we have

- (1') $\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{N}}(s) = c(1)$, $\mu([0, 1)) = 0$ and $\mu(\{1\}) = 1$.
- (2') $\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{R}\mathcal{R}}(s) = \infty$.
- (3') $\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{N}\mathcal{R}}(s) = \infty$, $\mu([0, 1)) = 0$ and $\mu(\{1\}) = 1$.
- (4') $\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{R}\mathcal{N}}(s) = c(1)$, $\mu([0, 1)) = 0$ and $\mu(\{1\}) = 1$.

We also note that, in Examples 2 and 3, $\lambda_1^{\mathcal{D}\mathcal{D}}(x_1, x_2) \rightarrow \infty$ and $\lambda_1^{\mathcal{N}\mathcal{D}}(x_1, x_2) \rightarrow \infty$ as x_1, x_2 shrinks to the point x_0 . Again, our above results are consistent with Theorem 1.1 due to Chen and Lou [3]; though $m'(x_0) = 0$, we do not require non-degeneracy of m at x_0 (that is, $m''(x_0)$ may vanish). Furthermore, in each of the above three examples, the support of μ consists of the set through which the limiting value of the principal eigenvalue is attained, and conversely, the limit is attained on the support of μ in the same sense as interpreted in Remark 2.1.

To end the introduction, we shall use two simple examples to hint the interesting impact of *oscillating behavior* of m on the principal eigenvalue.

Example A: Assume that the set of points of local maximum of m , denoted by \mathcal{M}_1^* , contains only isolated points, and x_0 is the only accumulation point of \mathcal{M}_1^* (that is, there is a sequence $\{x_i^I\}$ with $x_i^I \in \mathcal{M}_1^*$, $\forall i \geq 1$ such that $x_i^I \rightarrow x_0 \in (0, 1)$ as $i \rightarrow \infty$) and x_0 is also a point of local minimum of c (in the usual sense). Then, one can appeal to the analysis of this paper to show

$$\lim_{s \rightarrow \infty} \lambda_1(s) = \min\{c(x_0), \inf\{c(x), x \in \mathcal{M}_1^*\}\}.$$

Note that x_0 may not be an isolated point of local maximum of such a given m . This implies that such an oscillating behavior of m can affect the limiting profile of the principal eigenvalue $\lambda_1(s)$.

Example B: Assume that the set of points of local maximum of m is given by (1.9) in which we now take $h_2 = \infty, h_5 = \infty$ so that $a_i^D < x_0 < b_i^I$ for all $i \geq 1$, $a_i^I, b_i^I \rightarrow x_0 \in (0, 1)$ and

$a_i^D, b_i^D \rightarrow x_0$ as $i \rightarrow \infty$. Then the limit $\lim_{s \rightarrow \infty} \lambda_1(s)$ is given as in Theorem 1.2 with h_2, h_5 replaced by ∞ and so $\lim_{s \rightarrow \infty} \lambda_1(s)$ is independent of such x_0 . In other words, such an oscillating behavior of m has no qualitative effect on the limiting profile of the principal eigenvalue $\lambda_1(s)$.

Thus, it would be interesting to discuss the asymptotic behavior of the principal eigenvalue when the advection function m allows general oscillation.

The outline of this paper is as follows. Section 2 is devoted to the analysis of the limiting behavior of the principal eigenvalue and Theorems 1.2 and 1.3 are proved, while section 3 concerns the asymptotic profile of the principal eigenfunction in which Theorem 1.4 is verified. Throughout the paper, we use $|E|$ to stand for the Lebesgue measure of a given set $E \subset \mathbb{R}$.

2. THE PRINCIPAL EIGENVALUE: PROOF OF THEOREMS 1.2 AND 1.3

This section aims to analyze the asymptotic behavior of the principal eigenvalue of (1.1) and (1.6) as $s \rightarrow \infty$. We first consider the Neumann problem (1.2), and then investigate the general problem (1.1) and the periodic problem (1.6).

2.1. The principal eigenvalue problem (1.2). First of all, we estimate the upper bounds of $\lambda_1^N(s)$, and state that

Lemma 2.1. *Let $\lambda^* = \limsup_{s \rightarrow \infty} \lambda_1^N(s)$. The following assertions hold.*

- (i) If $x^I \in \mathcal{M}_1$, then $\lambda^* \leq c(x^I)$;
- (ii) If $[a^I, b^I] \in \mathcal{M}_2$, then $\lambda^* \leq \lambda_1^{ND}(a^I, b^I)$;
- (iii) If $[a^I, b^D] \in \mathcal{M}_3$, then $\lambda^* \leq \lambda_1^{NN}(a^I, b^D)$;
- (iv) If $[a^D, b^I] \in \mathcal{M}_4$, then $\lambda^* \leq \lambda_1^{DD}(a^D, b^I)$;
- (v) If $[a^D, b^D] \in \mathcal{M}_5$, then $\lambda^* \leq \lambda_1^{DN}(a^D, b^D)$;
- (vi) If $[0, a^I] \in \mathcal{M}_6$, then $\lambda^* \leq \lambda_1^{ND}(0, a^I)$;
- (vii) If $[0, a^D] \in \mathcal{M}_7$, then $\lambda^* \leq \lambda_1^{NN}(0, a^D)$;
- (viii) If $[a^I, 1] \in \mathcal{M}_8$, then $\lambda^* \leq \lambda_1^{ND}(a^I, 1)$;
- (ix) If $[a^D, 1] \in \mathcal{M}_9$, then $\lambda^* \leq \lambda_1^{DN}(a^D, 1)$.

Proof. The assertion (i) follows directly from Lemma 2.2 of [3]. In the sequel, we are going to verify the assertion (iii) by modifying the argument of Lemma 2.2 of [3]. Without loss of generality, we may assume that $c(x) \geq 0$ on $[0, 1]$; otherwise, we replace $(\lambda_1^N(s), c(x))$ in (1.2) by $(\lambda_1^N(s) + \max_{[0,1]} |c(x)|, c(x) + \max_{[0,1]} |c(x)|)$.

According to our definition for $[a^I, b^D]$, there exists a small $\epsilon_0 > 0$ such that m is constant on $[a^I, b^D]$, non-decreasing on $[a^I - \epsilon_0, a^I]$, and non-increasing on $[b^D, b^D + \epsilon_0]$. Furthermore, for any small $\epsilon > 0$, $\{x \in [0, 1] : m'(x) > 0\} \cap (a^I - \epsilon, a^I) \neq \emptyset$ and $\{x \in [0, 1] : m'(x) < 0\} \cap (b^D, b^D + \epsilon) \neq \emptyset$. We take φ_0 to be an principal eigenfunction corresponding to $\lambda_1^{NN}(a^I, b^D)$. Then, for any given three positive constant sequences $\{\alpha_i\}_{i=1}^\infty, \{\beta_i\}_{i=1}^\infty, \{\gamma_i\}_{i=1}^\infty$ satisfying $0 < \alpha_i < \beta_i < \gamma_i < \epsilon_0$ for each $i \geq 1$, and $\gamma_i \rightarrow 0$ (and so $\alpha_i, \beta_i \rightarrow 0$) as $i \rightarrow \infty$, we may assume, without loss of generality, that $m(a^I - \beta_i) - m(a^I - \alpha_i) < 0$ and $m(b^D + \beta_i) - m(b^D + \alpha_i) < 0$ for each $i \geq 1$.

We now choose the continuous function sequence $\{u_i\}_{i=1}^\infty$:

$$u_i(x) = \begin{cases} 0 & \text{if } x \in [0, a^I - \gamma_i], \\ \frac{\gamma_i - a^I + x}{\gamma_i - \beta_i} \varphi_0(a^I) & \text{if } x \in (a^I - \gamma_i, a^I - \beta_i], \\ \varphi_0(a^I) & \text{if } x \in (a^I - \beta_i, a^I], \\ \varphi_0(x) & \text{if } x \in (a^I, b^D], \\ \varphi_0(b^D) & \text{if } x \in (b^D, b^D + \beta_i], \\ \frac{\gamma_i + b^D - x}{\gamma_i - \beta_i} \varphi_0(b^D) & \text{if } x \in (b^D + \beta_i, b^D + \gamma_i], \\ 0 & \text{if } x \in (b^D + \gamma_i, 1]. \end{cases}$$

By the variational characterization for $\lambda_1(s)$ and $\lambda_1^{\mathcal{NN}}(a^I, b^D)$, elementary computation gives

$$\begin{aligned} \lambda_1^{\mathcal{N}}(s) &\leq \frac{\int_0^1 e^{2sm(x)} [(u'_i)^2 + c(x)u_i^2] dx}{\int_0^1 e^{2sm(x)} u_i^2 dx} \\ &\leq \frac{\int_{a^I}^{b^D} e^{2sm(x)} [(u'_i)^2 + c(x)u_i^2] dx}{\int_{a^I}^{b^D} e^{2sm(x)} u_i^2 dx} + \frac{\left(\int_{a^I - \gamma_i}^{a^I} + \int_{b^D}^{b^D + \gamma_i} \right) \left\{ e^{2sm(x)} [(u'_i)^2 + c(x)u_i^2] \right\} dx}{\int_0^1 e^{2sm(x)} u_i^2 dx} \\ &\leq \lambda_1^{\mathcal{NN}}(a^I, b^D) + I + II, \end{aligned}$$

where

$$I = \frac{\int_{a^I - \gamma_i}^{a^I} e^{2sm(x)} [(u'_i)^2 + c(x)u_i^2] dx}{\int_0^1 e^{2sm(x)} u_i^2 dx}, \quad II = \frac{\int_{b^D}^{b^D + \gamma_i} e^{2sm(x)} [(u'_i)^2 + c(x)u_i^2] dx}{\int_0^1 e^{2sm(x)} u_i^2 dx}.$$

Since $m(x)$ is non-decreasing in $[a^I - \gamma_i, a^I]$ for each $i \geq 1$ and $m(x)$ is constant on $[a^I, b^D]$, for any $s \geq 0$ we deduce

$$\begin{aligned} I &\leq \frac{\max_{[0,1]} |c(x)| \int_{a^I - \gamma_i}^{a^I} e^{2sm(x)} u_i^2 dx}{\int_{a^I}^{b^D} e^{2sm(x)} u_i^2 dx} + \frac{\int_{a^I - \gamma_i}^{a^I - \beta_i} e^{2sm(x)} (u'_i)^2 dx}{\int_{a^I - \alpha_i}^{a^I} e^{2sm(x)} u_i^2 dx} \\ &\leq \frac{\max_{[0,1]} |c(x)| \int_{a^I - \gamma_i}^{a^I} e^{2sm(a^I)} u_i^2 dx}{\int_{a^I}^{b^D} e^{2sm(a^I)} u_i^2 dx} + \frac{e^{2s[m(a^I - \beta_i) - m(a^I - \alpha_i)]}}{\alpha_i(\gamma_i - \beta_i)}. \end{aligned}$$

Due to $m(a^I - \beta_i) - m(a^I - \alpha_i) < 0$ for each $i \geq 1$, by sending $s \rightarrow \infty$ first and then sending $i \rightarrow \infty$, we easily see that $I \rightarrow 0$. Similarly, the term $II \rightarrow 0$ by sending $s \rightarrow \infty$ first and then sending $i \rightarrow \infty$. Consequently, $\lambda^* \leq \lambda_1^{\mathcal{NN}}(a^I, b^D)$, as wanted.

The remaining assertions can be proved in a similar way, and the details are omitted. \square

In order to estimate the lower bounds of $\lambda_1^{\mathcal{N}}(s)$, we need several key lemmas as follows. In what follows, let us remember that $w(s, \cdot)$ is the positive solution of (1.4) corresponding to the principal eigenvalue $\lambda_1^{\mathcal{N}}(s)$ with the normalization $\int_0^1 w^2(s, x) dx = 1$ for each $s > 0$. Let μ be the probability measure associated with $w(s_j, \cdot)$ defined through (1.10). Then we have

Lemma 2.2. *The following assertions hold.*

- (i) *Assume that there exists a closed interval $[a, b] \subset [0, 1]$ such that $\lim_{j \rightarrow \infty} \int_a^b w^2(s_j, x) dx \rightarrow 0$, then $\mu((a, b)) = 0$.*

- (ii) Assume that there exists a closed interval $[a, b] \subset [0, 1]$ such that $\mu([a, b]) = 0$, then $\lim_{j \rightarrow \infty} \int_a^b w^2(s_j, x) dx \rightarrow 0$.
- (iii) Assume that there exist two closed intervals $[a_*, a]$ and $[b, b_*]$ of $[0, 1]$ with $0 < a < b < 1$ such that $\mu([a_*, a]) = \mu([b, b_*]) = 0$, then $\lim_{j \rightarrow \infty} \int_a^b w^2(s_j, x) dx \rightarrow \mu((a, b))$.

Proof. The assertion (i) is trivial. Indeed, it is known that $\mu((a, b)) = \sup\{\mu([\tilde{a}, \tilde{b}]) : [\tilde{a}, \tilde{b}] \subset (a, b)\}$. Also, using (1.10), it easily follows $\mu([\tilde{a}, \tilde{b}]) = 0$ for any closed interval $[\tilde{a}, \tilde{b}] \subset (a, b)$. Thus, (i) holds.

We next prove (ii). Arguing indirectly, we suppose that there exist a constant $\epsilon_0 > 0$ and a subsequence of $\{s_j\}$, denoted by itself for convenience, such that

$$(2.1) \quad \int_a^b w^2(s_j, x) dx \geq \epsilon_0, \quad \forall j \geq 1.$$

By restricting $w^2(s_j, \cdot)$ to (a, b) , we may assume, up to a further subsequence, that

$$(2.2) \quad \lim_{j \rightarrow \infty} \int_a^b w^2(s_j, x) \zeta(x) dx = \int_{[a, b]} \zeta(x) \mu^*(dx), \quad \forall \zeta \in C([a, b])$$

for a unique Radon measure μ^* . Taking $\zeta = 1$ in (2.2) and using (2.1), we have $\mu^*([a, b]) \geq \epsilon_0$.

On the other hand, for each $j \geq 1$, we decompose $w^2(s_j, \cdot)$ as

$$(2.3) \quad w^2(s_j, \cdot) = w^2(s_j, \cdot) \chi_{[0, a]} + w^2(s_j, \cdot) \chi_{[a, b]} + w^2(s_j, \cdot) \chi_{(b, 1]},$$

where χ_X represents the characteristic function over a set $X \subset [0, 1]$. As above, we assume that

$$\int_0^1 w^2(s_j, x) \chi_{[0, a]} \zeta(x) dx \rightarrow \int_{[0, 1]} \zeta(x) \mu_1(dx), \quad \int_0^1 w^2(s_j, x) \chi_{[a, b]} \zeta(x) dx \rightarrow \int_{[0, 1]} \zeta(x) \mu_2(dx),$$

and

$$\int_0^1 w^2(s_j, x) \chi_{(b, 1]} \zeta(x) dx \rightarrow \int_{[0, 1]} \zeta(x) \mu_3(dx)$$

for any $\zeta \in C([0, 1])$, as $j \rightarrow \infty$, where μ_i ($i = 1, 2, 3$) are the certain Radon measures. In view of (2.3), it is easily seen from the definitions of μ, μ_i ($i = 1, 2, 3$) that $\mu = \mu_1 + \mu_2 + \mu_3$ on $[0, 1]$.

We further note that $\mu_2 = \mu^*$ on $[a, b]$. Indeed, since $w^2(s_j, \cdot) \chi_{[a, b]} = 0$ on $[0, a) \cup (b, 1]$, the analysis similar to that of the assertion (i) concludes that $\mu_2([0, a)) = \mu_2((b, 1]) = 0$. In addition,

$$\int_0^1 w^2(s_j, x) \chi_{[a, b]} \zeta(x) dx = \int_a^b w^2(s_j, x) \zeta(x) dx, \quad \forall \zeta \in C([a, b]).$$

Hereafter, we extend ζ , which is defined on $[a, b]$, continuously to $[0, 1]$ so that the integral over $[0, 1]$ makes sense. So it follows

$$\begin{aligned} \int_{[a, b]} \zeta(x) \mu^*(dx) &= \int_{[0, 1]} \zeta(x) \mu_2(dx) = \int_{[0, a)} \zeta(x) \mu_2(dx) + \int_{[a, b]} \zeta(x) \mu_2(dx) \\ &\quad + \int_{(b, 1]} \zeta(x) \mu_2(dx) = \int_{[a, b]} \zeta(x) \mu_2(dx), \quad \forall \zeta \in C([a, b]), \end{aligned}$$

which obviously implies $\mu_2 = \mu^*$ on $[a, b]$.

Therefore, we obtain that

$$\mu([a, b]) = \mu_1([a, b]) + \mu_2([a, b]) + \mu_3([a, b]) = \mu_1([a, b]) + \mu^*([a, b]) + \mu_3([a, b]) \geq \mu^*([a, b]) \geq \epsilon_0,$$

which arrives at a contradiction with our assumption $\mu([a, b]) = 0$. Thus, (ii) is proved.

Lastly, we verify (iii). In (1.10), we choose $\zeta = 1$ on $[a, b]$, $\zeta = 0$ on $[0, a_*] \cup [b_*, 1]$, and $0 \leq \zeta \leq 1$ in $[a_*, a] \cup [b, b_*]$ so that $\zeta \in C([0, 1])$. Thanks to $\mu([a_*, a]) = \mu([b, b_*]) = 0$, it then follows from the assertion (ii) that

$$\lim_{j \rightarrow \infty} \int_0^1 w^2(s_j, x) \zeta(x) dx = \lim_{j \rightarrow \infty} \int_a^b w^2(s_j, x) dx.$$

On the other hand, for such chosen ζ , because of $\mu([a_*, a]) = \mu([b, b_*]) = 0$, we have

$$\int_{[0, 1]} \zeta(x) \mu(dx) = \int_{(a, b)} \zeta(x) \mu(dx) = \int_{(a, b)} \mu(dx) = \mu((a, b)).$$

In light of (1.10), the desired conclusion is established. \square

Lemma 2.3. *The following assertions hold.*

(i) *Given any $x \in (0, 1)$ and any ϵ with $0 < \epsilon < \frac{1}{2} \min\{x, 1 - x\}$, then*

$$\lim_{j \rightarrow \infty} \int_{x-2\epsilon}^{x+2\epsilon} w^2(s_j, x) c(x) dx \geq \min_{[x-\epsilon, x+\epsilon]} c(x) \mu(\{x\}).$$

(ii) *Given any ϵ with $0 < \epsilon < 1/2$, then*

$$\lim_{j \rightarrow \infty} \int_0^{2\epsilon} w^2(s_j, x) c(x) dx \geq \min_{[0, \epsilon]} c(x) \mu(\{0\}).$$

(iii) *Given any ϵ with $0 < \epsilon < 1/2$ such that*

$$\lim_{j \rightarrow \infty} \int_{1-2\epsilon}^1 w^2(s_j, x) c(x) dx \geq \min_{[1-\epsilon, 1]} c(x) \mu(\{1\}).$$

Proof. This lemma is a straightforward consequence of (1.10) and (1.11). \square

The following preliminary results aim to give a precise description of the support of μ .

Lemma 2.4. *Denote $\Omega_1 = \{x \in (0, 1) : |m'(x)| > 0\} \cup \{0 : m'(0) > 0\} \cup \{1 : m'(1) < 0\}$. Then $\mu(\Omega_1) = 0$.*

Proof. This result follows directly from Lemma 3.1 and Lemma 3.4 of [3]. \square

Lemma 2.5. *The following assertions hold.*

- (i) *Assume that $|m'(x)| > 0$ in an open interval $(a, b) \subset [0, 1]$. Then $w(s, \cdot) \rightarrow 0$ locally uniformly in (a, b) as $s \rightarrow \infty$;*
- (ii) *Assume that $m'(x) > 0$ in an interval $[0, a) \subset [0, 1]$. Then $w(s, \cdot) \rightarrow 0$ locally uniformly in $[0, a)$ as $s \rightarrow \infty$.*
- (iii) *Assume that $m'(x) < 0$ in an interval $(a, 1] \subset [0, 1]$. Then $w(s, \cdot) \rightarrow 0$ locally uniformly in $(a, 1]$ as $s \rightarrow \infty$.*

Proof. Under the assumption of (i), we first claim that, by passing to a sequence, $w(s_j, \cdot) \rightarrow 0$ a.e. in (a, b) as $j \rightarrow \infty$. In fact, given small $\delta > 0$, we take $\zeta \in C([0, 1])$ in (1.10) such that $\zeta(x) = 1$ on $[a + 2\delta, b - 2\delta]$, $\zeta(x) = 0$ on $[0, a + \delta] \cup [b - \delta, 1]$ and $0 \leq \zeta(x) \leq 1$ on $[0, 1]$. Then from (1.10) and Lemma 2.4 it follows that

$$\begin{aligned} 0 \leq \lim_{j \rightarrow \infty} \int_{a+2\delta}^{b-2\delta} w^2(s_j, x) dx &\leq \lim_{j \rightarrow \infty} \int_{a+\delta}^{b-\delta} w^2(s_j, x) \zeta(x) dx \\ &= \int_{[a+\delta, b-\delta]} \zeta(x) \mu(dx) \leq \mu([a + \delta, b - \delta]) = 0. \end{aligned}$$

Clearly, this implies that $w(s_j, \cdot) \rightarrow 0$ a.e. in $(a + 2\delta, b - 2\delta)$ as $j \rightarrow \infty$. Since δ can be arbitrarily small, we have $w(s_j, \cdot) \rightarrow 0$ a.e. in (a, b) , as claimed.

Given small $\delta > 0$, by our assumption, there exists a positive constant $c_0 = c_0(\delta) < 1$ such that $|m'(x)| \geq c_0$ and $|m''(x)| \leq 1/c_0$ on $[a + \delta, b - \delta]$. Thanks to the claim proved before, we assume, without loss of generality, that $w(s_j, a + \delta) \rightarrow 0$ and $w(s_j, b - \delta) \rightarrow 0$ as $j \rightarrow \infty$. Thus, we can find a large integer J_0 such that $w(s_j, a + \delta), w(s_j, b - \delta) \leq 1, \forall j \geq J_0$. Furthermore, taking larger J_0 if necessary and using (1.4), we observe that, for all $j \geq J_0$, $w(s_j, x)$ satisfies

$$-w''(s_j, x) \leq -\frac{1}{2}c_0^2 s_j^2 w(s_j, x), \quad a + \delta < x < b - \delta.$$

Hence, for each $j \geq J_0$, $w(s_j, x)$ is a subsolution of the elliptic problem:

$$(2.4) \quad -u'' = -\frac{1}{2}c_0^2 s_j^2 u, \quad a + \delta < x < b - \delta; \quad u(s_j, a + \delta) = u(s_j, b - \delta) = 1.$$

Simple analysis shows that the unique solution of (2.4), denoted by u_j , satisfies $w(s_j, \cdot) \leq u_j \rightarrow 0$ locally uniformly in $(a + \delta, b - \delta)$ as $j \rightarrow \infty$. Due to the arbitrariness of δ , $w(s_j, \cdot) \rightarrow 0$ locally uniformly in (a, b) , and so the assertion (i) holds.

The assertions (ii) and (iii) can be proved similarly. Indeed, to verify (ii), since $m'(0), w(s_j, 0) > 0$, we get from the boundary condition in (1.4) that $w'(s_j, 0) > 0$ for each $j \geq 1$. As above, for any given small $\delta > 0$, there exist a positive constant $c_0 < 1$ and a large integer J_0 such that, for all $j \geq J_0$, $w(s_j, x)$ satisfies

$$-w''(s_j, x) \leq -\frac{1}{2}c_0^2 s_j^2 w(s_j, x), \quad 0 < x < a - \delta; \quad w'(s_j, 0) > 0, \quad w(s_j, a - \delta) \leq 1.$$

On the other hand, instead of considering the auxiliary problem (2.4), we resort to the problem:

$$(2.5) \quad -u'' = -\frac{1}{2}c_0^2 s_j^2 u, \quad 0 < x < a - \delta; \quad u'(s_j, 0) = 0, \quad u(s_j, a - \delta) = 1.$$

It is easy to check that the unique solution u_j of (2.5) converges to 0 locally uniformly in $[0, a - \delta)$. Clearly, $w(s_j, x)$ is a subsolution of (2.5). A simple comparison argument asserts $w(s_j, \cdot) \leq u_j$, and so $w(s_j, \cdot) \rightarrow 0$ locally uniformly in $[0, a)$ as $j \rightarrow \infty$. Thus, Lemma 2.5 is proved. \square

Lemma 2.6. *Assume that m is strictly increasing or strictly decreasing on $[a_1, a_2] \subset [0, 1]$, then $\mu((a_1, a_2)) = 0$.*

Proof. We only consider the case that m is strictly increasing on $[a_1, a_2]$ since the assertion in the other case can be established similarly. To the end, it suffices to show that $\mu((a_1, b_i]) = 0, \forall i \geq 1$ for a given sequence $\{b_i\}_{i \geq 1}$ satisfying $b_i < a_2$ for each $i \geq 1$ and $b_i \rightarrow a_2$ as $i \rightarrow \infty$.

For any fixed $i \geq 1$, we first claim that

$$(2.6) \quad \|w(s_j, \cdot)\|_{L^\infty(a_1, b_i)} \leq M \text{ for some positive constant } M, \text{ independent of } j \geq 1.$$

We proceed by an indirect argument. To produce a contradiction, the analysis below turns out to be rather long, and for clarity we divide it into three steps.

Step 1. Suppose that the claim (2.6) is false. Then, there is a sequence of $\{s_j\}_{j \geq 1}$, labelled by itself for convenience, such that $\|w(s_j, \cdot)\|_{L^\infty(a_1, b_i)} \rightarrow \infty$ as $j \rightarrow \infty$. For each j , we can find $x_j \in [a_1, b_i]$ such that $w(s_j, x_j) \rightarrow \infty$ as $j \rightarrow \infty$. By taking

$$M = 2 \left(1 + \max_{[0,1]} |c(x)| \right)^{1/2},$$

we have $w(s_j, x_j) \geq M$, $\forall j \geq j_0$ for some large integer j_0 .

On the other hand, as m is strictly increasing on $[a_1, a_2]$, it is clear that $\{m'(x) > 0\} \cap (b_i, a_2) \neq \emptyset$. So one can find a $y_0 \in (b_i, a_2)$ such that $m'(x) > 0$ in a small neighbourhood of y_0 . Hence, Lemma 2.5 ensures that, by taking larger j_0 if necessary, $w(s_j, y_0) \leq 1/M$ for all $j \geq j_0$.

From now on, we fix $j = j_0$. For simplicity, denote

$$a_* = x_{j_0}, \quad b_* = y_0 \quad \text{and} \quad w(x) = w(s_{j_0}, x).$$

Thus, $a_* < b_*$, and

$$(2.7) \quad w(b_*) \leq 1/M < M \leq w(a_*).$$

In view of $w \in C^1([a_*, b_*])$ and (2.7), clearly, $\{x \in (a_*, b_*) : w'(x) < 0\}$ is a nonempty open set, which is therefore the union of at most countably many disjoint open intervals and

$$\mathcal{G} := \{x \in [a_*, b_*] : w'(x) = 0\}$$

is a closed set. So we assume that

$$\{x \in (a_*, b_*) : w'(x) < 0\} = \bigcup_{i \in \mathbb{N}} (\hat{a}_i, \hat{b}_i),$$

where \mathbb{N} is a given set consisting of at most countably many integers, and $(\hat{a}_i, \hat{b}_i) \cap (\hat{a}_j, \hat{b}_j) = \emptyset$, $\forall i \neq j$. In addition, since w is strictly decreasing in each (\hat{a}_i, \hat{b}_i) , it is easily seen that $(w(\hat{b}_i), w(\hat{a}_i))$ is an open interval and

$$(2.8) \quad (w(b_*), w(a_*)) \subseteq w(\mathcal{G}) \cup \bigcup_{i \in \mathbb{N}} (w(\hat{b}_i), w(\hat{a}_i)).$$

From Sard's Lemma (see, for instance, Theorem 3.6.3 of [1]) it follows that the Lebesgue measure $|w(\mathcal{G})| = 0$. This fact, together with (2.8), allows us to assert that

$$(2.9) \quad \left| \bigcup_{i \in \mathbb{N}} (w(\hat{b}_i), w(\hat{a}_i)) \right| = \left| w(\mathcal{G}) \cup \bigcup_{i \in \mathbb{N}} (w(\hat{b}_i), w(\hat{a}_i)) \right| \geq w(a_*) - w(b_*).$$

Step 2. The rearrangement of the curve sequence $\{(x, w(x)) : x \in (\hat{a}_i, \hat{b}_i)\}_{i \in \mathbb{N}}$. We will proceed in three substeps. Set

$$\mathcal{F} = \{(\hat{a}_i, \hat{b}_i) : i \in \mathbb{N}\}.$$

Recall that w is strictly decreasing in each $(\hat{a}_i, \hat{b}_i) \in \mathcal{F}$. We take an arbitrary interval, say, $(\hat{a}_1, \hat{b}_1) \in \mathcal{F}$.

As a first substep, we are going to do the upward extension for the C^1 -curve $\{(x, w(x)) : x \in [\hat{a}_1, \hat{b}_1]\}$ by picking up some other elements from the curve sequence $\{(x, w(x)) : x \in (\hat{a}_i, \hat{b}_i)\}_{i \in \mathbb{N}}$ appropriately. We start at the highest point $(\hat{a}_1, w(\hat{a}_1))$ of $\{(x, w(x)) : x \in [\hat{a}_1, \hat{b}_1]\}$, and operate in the following procedures.

If $w(\hat{a}_1) \geq w(x)$ for all $x \in [\hat{a}_i, \hat{b}_i]$ and any $i \in \mathbb{N}$, then we do not need conduct the upward extension, and just define $\underline{w}_1(x) = w(x)$ on $[\hat{a}_1, \hat{b}_1]$.

If there exists some $(\underline{a}, \underline{b}) \in \mathcal{F}$ satisfying $w(\hat{a}_1) \in [w(\underline{b}), w(\underline{a})]$, then there is a unique $\underline{c} \in (\underline{a}, \underline{b}]$ such that $w(\hat{a}_1) = w(\underline{c}) < w(x)$ for all $x \in [\underline{a}, \underline{c})$. Then, we move the curve $\{(x, w(x)) : x \in [\underline{a}, \underline{c}]\}$ by horizontal translation so that its lowest point $(\underline{c}, w(\underline{c}))$ overlaps the highest point $(\hat{a}_1, w(\hat{a}_1))$ of the curve $\{(x, w(x)) : x \in [\hat{a}_1, \hat{b}_1]\}$. Thus, we obtain an extended curve, denoted by $\{(x, \underline{w}_1(x)) : x \in [\underline{a}_1, \hat{b}_1]\}$, with the highest point $(\underline{a}_1, w(\underline{a}))$, where $\underline{a}_1 = \hat{a}_1 + \underline{a} - \underline{c}$ and \underline{w}_1 is a continuous and strictly decreasing function on $[\underline{a}_1, \hat{b}_1]$.

Now, if there exists some $(\tilde{a}, \tilde{b}) \in \mathcal{F}$ satisfying $w_1(\underline{a}_1) \in [w(\tilde{b}), w(\tilde{a})]$, then there is a unique $\tilde{c} \in (\tilde{a}, \tilde{b}]$ such that $w_1(\underline{a}_1) = w(\tilde{c}) < w(x)$ for all $x \in [\tilde{a}, \tilde{c})$. Similarly as before, translating the curve $\{(x, w(x)) : x \in [\tilde{a}, \tilde{c}]\}$ until its lowest point $(\tilde{c}, w(\tilde{c}))$ coincides with the highest point $(\underline{a}_1, w_1(\underline{a}_1))$ of the curve $\{(x, w_1(x)) : x \in [\underline{a}_1, \hat{b}_1]\}$, we obtain an extended continuous curve which has the highest point $(\underline{a}_2, w(\tilde{a}))$, where $\underline{a}_2 = \underline{a}_1 + \tilde{a} - \tilde{c}$. We use \underline{w}_2 , which is defined on $[\underline{a}_2, \hat{b}_1]$ and continuous and strictly decreasing, to represent the function of such a new curve.

If such $(\tilde{a}, \tilde{b}) \in \mathcal{F}$ can not be found, we then define $\underline{w}_2(x) = \underline{w}_1(x)$ on $[\underline{a}_1, \hat{b}_1]$.

In a similar way, we conduct the upward extension for the curve $\{(x, \underline{w}_2(x)) : x \in [\underline{a}_2, \hat{b}_1]\}$. After at most countably many times, we obtain the continuous curve $\{(x, \underline{w}_1(x)) : x \in [a_{1,+}, \hat{b}_1]\}$, where $a_{1,+} = \inf\{a_0, \underline{a}_1, \underline{a}_2, \dots\} \geq \hat{b}_1 - (b_* - a_*)$, and $\underline{W}_1(x) = \underline{w}_k(x)$, $\forall x \in [\underline{a}_k, \hat{b}_1]$, $k = 0, 1, 2, \dots$, and $\underline{W}_1(a_{1,+}) \notin [w(\hat{b}_1), w(\hat{a}_1)]$, $\forall (\hat{a}_i, \hat{b}_i) \in \mathcal{F}$. Here $\underline{a}_0 = \hat{a}_1$. Note that a continuous extension from right was made here at $x = a_{1,+}$ for \underline{W}_1 so that $\underline{W}_1 \in C([a_{1,+}, \hat{b}_1])$ if the above procedures are of infinite times. So far, we have finished the upward extension of the original curve $\{(x, w(x)) : x \in [\hat{a}_1, \hat{b}_1]\}$.

Secondly, we carry out the downward extension for $\{(x, \underline{W}_1(x)) : x \in [a_{1,+}, \hat{b}_1]\}$, starting at its lowest point $(\hat{b}_1, \underline{W}_1(\hat{b}_1))$. The procedures are similar to those of the upward extension:

If we can find some $(\bar{a}, \bar{b}) \in \mathcal{F}$ satisfying $\underline{W}_1(\hat{b}_1) \in (w(\bar{b}), w(\bar{a})]$, then there is a unique $\bar{c} \in [\bar{a}, \bar{b})$ such that $\underline{W}_1(\hat{b}_1) = w(\bar{c}) > w(x)$, $\forall x \in (\bar{c}, \bar{b}]$. Again, horizontally one can translate the curve $\{(x, w(x)) : x \in [\bar{c}, \bar{b}]\}$ until its highest point $(\bar{c}, w(\bar{c}))$ overlaps the lowest point $(\hat{b}_1, \underline{W}_1(\hat{b}_1))$ of $\{(x, \underline{W}_1(x)) : x \in [a_{1,+}, \hat{b}_1]\}$. Denote by $\{(x, \bar{W}_1(x)) : x \in [a_{1,+}, \bar{b}_1]\}$ such an extended curve. Then, $0 < \bar{b}_1 - \hat{b}_1 = \bar{b} - \bar{c}$ and \bar{W}_1 is a strictly decreasing continuous function on $[a_{1,+}, \bar{b}_1]$.

If such $(\bar{a}, \bar{b}) \in \mathcal{F}$ does not exist, we then define $\bar{W}_1(x) = \underline{W}_1(x)$ on $[a_{1,+}, \hat{b}_1]$.

Set $\bar{b}_0 = \hat{b}_1$. Proceeding similarly as above, we do the possible downward extension for the curve $\{(x, \bar{W}_1(x)) : x \in [a_{1,+}, \bar{a}_1]\}$. After at most countably many times, we can obtain the curve $\{(x, W_1(x)) : x \in [a_{1,+}, b_1^+]\}$, where $b_1^+ = \sup\{\bar{b}_0, \bar{b}_1, \bar{b}_2, \dots\}$ satisfying $0 < b_1^+ - a_{1,+} \leq b_* - a_*$, and $W_1 \in C([a_{1,+}, b_1^+])$ enjoys the properties: $W_1(x) = \bar{W}_k(x)$, $\forall x \in [a_{1,+}, \bar{b}_k]$, $k = 0, 1, 2, \dots$, W_1 is strictly decreasing on $[a_{1,+}, b_1^+]$, and

$$W_1(a_{1,+}) \notin [w(\hat{b}_i), w(\hat{a}_i)], \quad W_1(b_1^+) \notin (w(\hat{b}_i), w(\hat{a}_i)], \quad \forall (\hat{a}_i, \hat{b}_i) \in \mathcal{F}.$$

This finishes the maximal extension of the curve $\{(x, w(x)) : x \in [\hat{a}_1, \hat{b}_1]\}$.

Thirdly, if W_1 satisfies $W_1(a_{1,+}) - W_1(b_1^+) \geq w(a_*) - w(b_*)$, we stop. If $W_1(a_{1,+}) - W_1(b_1^+) < w(a_*) - w(b_*)$, by (2.9) and the extension conducted above, it is easy to see that there is an interval, say, $(\hat{a}_2, \hat{b}_2) \in \mathcal{F}$ such that $[w(\hat{b}_2), w(\hat{a}_2)] \cap [W_1(b_1^+), W_1(a_{1,+})] = \emptyset$. In such a situation, following the same procedures as before, we extend the curve $\{(x, w(x)) : x \in [\hat{a}_2, \hat{b}_2]\}$ to a maximal one, whose function W_2 , defined on an interval, say $[a_{2,+}, b_2^+] \supset [\hat{a}_2, \hat{b}_2]$, is continuous and strictly decreasing. Moreover, $\sum_{i=1}^2 (b_i^+ - a_{i,+}) \leq b_* - a_*$, and

$$W_2(a_{2,+}) \notin [w(\hat{b}_2), w(\hat{a}_2)], \quad W_2(b_2^+) \notin (w(\hat{b}_2), w(\hat{a}_2)], \quad \forall (\hat{a}_2, \hat{b}_2) \in \mathcal{F}.$$

If

$$W_1(a_{1,+}) - W_1(b_1^+) + W_2(a_{2,+}) - W_2(b_2^+) \geq w(a_*) - w(b_*),$$

we stop; if

$$W_1(a_{1,+}) - W_1(b_1^+) + W_2(a_{2,+}) - W_2(b_2^+) < w(a_*) - w(b_*),$$

arguing as above, we can find an interval, say, $(\hat{a}_3, \hat{b}_3) \in \mathcal{F}$ such that

$$[w(\hat{b}_3), w(\hat{a}_3)] \cap \bigcup_{i=1}^2 ([W_i(b_i^+), W_i(a_{i,+})]) = \emptyset,$$

and then we extend the curve $\{(x, w(x)) : x \in [\hat{a}_3, \hat{b}_3]\}$ to a maximal one, whose function W_3 , defined on an interval, say $[a_{3,+}, b_3^+] \supset [\hat{a}_3, \hat{b}_3]$, is continuous and strictly decreasing, and

$$\sum_{i=1}^3 (b_i^+ - a_{i,+}) \leq b_* - a_*, \quad W_3(a_{3,+}) \notin [w(\hat{b}_3), w(\hat{a}_3)], \quad W_3(b_3^+) \notin (w(\hat{b}_3), w(\hat{a}_3)], \quad \forall (\hat{a}_i, \hat{b}_i) \in \mathcal{F}.$$

Up to at most countably many times, we obtain the function sequence $\{W_i : i \in \mathbb{E}\}$ with \mathbb{E} being a given set consisting of at most countably many integers, which satisfies

- (p1) Each W_i is continuous, and strictly decreasing on its domain $[a_{i,+}, b_i^+]$ with $\sum_{i \in \mathbb{E}} (b_i^+ - a_{i,+}) \leq b_* - a_*$, and is $C^1([a_{i,+}, b_i^+] \setminus \mathcal{O}_i)$ with each \mathcal{O}_i containing at most countably many points;
- (p2) $(W_i(b_i^+), W_i(a_{i,+})) \cap (W_j(b_j^+), W_j(a_{j,+})) = \emptyset$, $\forall i \neq j$, and $W_j(a_{j,+}) \notin [w(\hat{b}_i), w(\hat{a}_i)]$, $W_j(b_j^+) \notin (w(\hat{b}_i), w(\hat{a}_i)]$, $\forall j \in \mathbb{E}$, $\forall (\hat{a}_i, \hat{b}_i) \in \mathcal{F}$;
- (p3) For any $i \in \mathbb{N}$ and any $x \in (\hat{a}_i, \hat{b}_i) \in \mathcal{F}$, there is a W_j such that $w(x) \in [W_j(b_j^+), W_j(a_{j,+})]$;
- (p4) $\sum_{i \in \mathbb{E}} \int_{a_{i,+}}^{b_i^+} |W'_i(x)|^2 dx \leq \sum_{i \in \mathbb{N}} \int_{\hat{a}_i}^{\hat{b}_i} |w'(x)|^2 dx \leq \int_{a_*}^{b_*} |w'(x)|^2 dx$.

Hence, from (p3) and (2.9), it immediately follows that

$$(2.10) \quad \left| \bigcup_{i \in \mathbb{E}} [W_i(b_i^+), W_i(a_{i,+})] \right| \geq \left| \bigcup_{i \in \mathbb{N}} (w(\hat{b}_i), w(\hat{a}_i)) \right| \geq w(a_*) - w(b_*).$$

Finally, in view of (p1)-(p4), through at most countably many times of translation transformations (including possible horizontal and vertical translations) over the curve sequence $\{(x, W_i(x)) : x \in [a_{i,+}, b_i^+], i \in \mathbb{E}\}$, we can get a function W , which has the following properties:

- W is strictly decreasing on its domain $[a_\infty, b_\infty]$ with $0 < b_\infty - a_\infty \leq b_* - a_*$, is continuous at a_∞ and b_∞ , and is $C^1([a_\infty, b_\infty] \setminus \mathcal{O})$ with \mathcal{O} containing at most countably many points;
- $\left| \bigcup_{i \in \mathbb{E}} [W_i(b_i^+), W_i(a_{i,+})] \right| = W(a_\infty) - W(b_\infty)$, which, combined with (2.10), implies that $W(a_\infty) - W(b_\infty) \geq w(a_*) - w(b_*)$;

$$\bullet \int_{a_\infty}^{b_\infty} |W'(x)|^2 dx = \sum_{i \in \mathbb{E}} \int_{a_{i,+}}^{b_{i,+}} |W'_i(x)|^2 dx \leq \sum_{i \in \mathbb{N}} \int_{\hat{a}_i}^{\hat{b}_i} |w'(x)|^2 dx \leq \int_{a_*}^{b_*} |w'(x)|^2 dx.$$

Step 3. We use the same notation as in step 2. Based on what was proved by step 2, together with the fact of $m' \geq 0$ and $w'(x) \leq 0$ on each $[\hat{a}_i, \hat{b}_i] \subset [0, 1]$, $\forall i \in \mathbb{N}$, we deduce from (1.5) that

$$(2.11) \quad \begin{aligned} \max_{[0,1]} |c(x)| &\geq \lambda^* \geq \int_{a_*}^{b_*} (w'(x) - sw(x)m'(x))^2 dx - \max_{[0,1]} |c(x)| \\ &\geq \sum_{i \in \mathbb{N}} \int_{\hat{a}_i}^{\hat{b}_i} |w'(x)|^2 dx - \max_{[0,1]} |c(x)| \geq \int_{a_\infty}^{b_\infty} |W'(x)|^2 dx - \max_{[0,1]} |c(x)|. \end{aligned}$$

We then aim to estimate the integral $\int_{a_\infty}^{b_\infty} |W'(x)|^2 dx$. It is well known that $H^1((a_\infty, b_\infty))$ is compactly embedded into $C([a_\infty, b_\infty])$. Let us consider the minimizer of the functional

$$\int_{a_\infty}^{b_\infty} |u'(x)|^2 dx, \quad \forall u \in H^1(a_\infty, b_\infty),$$

under the constrained conditions $u(a_\infty) = W(a_\infty)$, $u(b_\infty) = W(b_\infty)$. It is easy to check that the minimizer of such a functional is attainable and its minimal u_0 is a solution of the following ODE problem with two-point boundary values

$$u_0'' = 0, \quad x \in (a_\infty, b_\infty); \quad u_0(a_\infty) = W(a_\infty), \quad u_0(b_\infty) = W(b_\infty).$$

Thus, u_0 is the segment connecting the two endpoints a_∞ and b_∞ . So we have

$$\int_{a_\infty}^{b_\infty} |W'(x)|^2 dx \geq \int_{a_\infty}^{b_\infty} |u_0'(x)|^2 dx = \frac{|W(a_\infty) - W(b_\infty)|^2}{b_\infty - a_\infty}.$$

Recall that $0 < b_\infty - a_\infty \leq b_* - a_* \leq 1$ and $W(a_\infty) - W(b_\infty) \geq w(a_*) - w(b_*)$. As a consequence, this, together with (2.7) and (2.11), yields

$$2 \max_{[0,1]} |c(x)| \geq \frac{|w(a_*) - w(b_*)|^2}{b_* - a_*} \geq \frac{1}{2} \left(M - \frac{1}{M} \right)^2,$$

which leads to an obvious contraction due to the choice of M . Therefore, the claim (2.6) is proved.

As m is strictly increasing on $[a_1, a_2]$, we know that $m'(x) \geq 0$ on $[a_1, a_2]$. Set $\mathcal{C} = \{x \in [a_1, a_2] : m'(x) = 0\}$. In the sequel, we need consider two different cases: Case **A**: $|\mathcal{C}| = 0$; Case **B**: $|\mathcal{C}| > 0$.

We first treat Case **A**: $|\mathcal{C}| = 0$. For any small $\delta > 0$, we take $\zeta(x) = 1$ on $[a_1 + \delta, b_i - \delta]$, $\zeta(x) = 0$ on $[0, a_1 + \frac{1}{2}\delta] \cup [b_i - \frac{1}{2}\delta, 1]$ and $0 \leq \zeta(x) \leq 1$ on $[a_1 + \frac{1}{2}\delta, a_1 + \delta] \cup [b_i - \delta, b_i - \frac{1}{2}\delta]$ so that $\zeta \in C([0, 1])$. By Lemma 2.5, we know that $w(s_j, \cdot) \rightarrow 0$ a.e. in $\{m'(x) > 0\}$. Hence, combined with this fact, $|\mathcal{C}| = 0$ and the Lebesgue's dominated convergence theorem, we can easily conclude from (1.10) and (2.6) that

$$\mu([a_1 + \delta, b_i - \delta]) \leq \lim_{j \rightarrow \infty} \int_{(a_1 + \delta/2, b_i - \delta/2) \cap \mathcal{C}} w^2(s_j, x) dx + \lim_{j \rightarrow \infty} \int_{\{m'(x) > 0\}} w^2(s_j, x) dx = 0,$$

which therefore gives $\mu((a_1, b_i)) = 0$, $\forall i \geq 1$ due to the arbitrariness of δ , and in turn $\mu((a_1, a_2)) = 0$ as $b_i \rightarrow a_2$.

We next consider Case **B**: $|\mathcal{C}| > 0$. Recall that m is strictly increasing on $[a_1, a_2]$. Thus, without loss of generality, we may assume that $m'(b_i) > 0, \forall i \geq 1$. In what follows, we fix $i \geq 1$. Then $m'(x) > 0$ on $(b_i - \epsilon_0/2, b_i)$ for some small $\epsilon_0 > 0$.

Set

$$\mathcal{A} = \{x \in (0, 1) : m'(x) > 0\}.$$

Since \mathcal{A} is a bounded open set, it is a union of at most countably many disjoint open intervals. So we can find a sequence of closed sets, say $\{\mathcal{A}_k\}_{k=1}^\infty$ such that $\mathcal{A}_k \subset \mathcal{A}, \mathcal{A}_k \subset \mathcal{A}_{k+1}, k \geq 1, \bigcup_{k=1}^\infty \mathcal{A}_k = \mathcal{A}$. Hence, given $k \geq 1$, for any small $\delta = \delta(k) > 0$, there holds

$$(2.12) \quad w(s_j, x) \leq \delta \text{ for all } x \in \mathcal{A}_k \text{ and for all large } j \text{ (due to Lemma 2.5).}$$

We now assert that for any given $0 < \epsilon \leq \epsilon_0/2$, there is an integer k_0 such that

$$(2.13) \quad d(x, \mathcal{A}_k \cap (x, b_i)) < \epsilon, \quad \forall x \in \{y \in [0, 1] : m'(y) = 0\} \cap (a_1, b_i), \quad \text{for all } k \geq k_0.$$

Here, $d(x, \mathcal{A}_k \cap (x, b_i))$ stands for the usual distance between the point x and the set $\mathcal{A}_k \cap (x, b_i)$. Indeed, if (2.13) does not hold, there is a subsequence $\{k_l\}_{l=1}^\infty$ with $k_l \rightarrow \infty$ as $l \rightarrow \infty$ and a point sequence $x_l \in \{y \in [0, 1] : m'(y) = 0\} \cap (a_1, b_i)$ such that

$$d(x_l, \mathcal{A}_{k_l} \cap (x_l, b_i)) \geq \epsilon_1, \quad \forall l \geq 1 \quad \text{for some } 0 < \epsilon_1 \leq \epsilon_0/2.$$

If $\mathcal{A}_{k_l} \cap (x_l, b_i) = \emptyset$ for some l , we define $d(x_l, \mathcal{A}_{k_l} \cap (x_l, b_i)) = \infty$. As $x_l \in \{y \in [0, 1] : m'(y) = 0\} \cap (a_1, b_i)$ and $m'(x) > 0$ on $(b_i - \epsilon_0/2, b_i)$, it is clear that $x_l \leq b_i - \epsilon_0$. Passing up to a subsequence, we assume that $x_l \rightarrow x^*$ and so $x^* \leq b_i - \epsilon_0$. Moreover, in view of the fact that $\mathcal{A}_{k_l} \subset \mathcal{A}_{k_l+1}, \forall l \geq 1$ and $\bigcup_{l=1}^\infty \mathcal{A}_{k_l} = \mathcal{A}$, by sending $l \rightarrow \infty$ it easily follows that

$$d(x^*, \mathcal{A} \cap (x^*, b_i)) \geq \epsilon_1,$$

which immediately implies that m is constant on $[x^*, x^* + \epsilon_1]$, an obvious contradiction! Hence, (2.13) holds.

We then conclude that

$$(2.14) \quad \limsup_{j \rightarrow \infty} \|w(s_j, \cdot)\|_{L^\infty((a_1, b_i) \setminus \mathcal{A})} := m^* = 0.$$

Once the assertion (2.14) holds, the argument in Case **A** can be easily adapted to show that $\mu((a_1, b_i)) = 0, \forall i \geq 1$ and in turn $\mu((a_1, a_2)) = 0$.

It remains to prove (2.14). Suppose that $m^* > 0$. Then we can find a sequence $x_j \in (a_1, b_i) \cap \{m'(x) = 0\}$ such that $w(s_j, x_j) \geq m^*/2, \forall j \geq j_0$ for some large j_0 . On the other hand, by taking $\delta = m^*/4$ and $\epsilon = \min\{\frac{1}{2}\epsilon_0, \frac{(m^*)^2}{64 \max_{[0,1]} |c(x)|}\}$ in (2.12) and (2.13), respectively, we see that there exists y_j with $0 < y_j - x_j < \epsilon$ such that $w(s_j, y_j) < m^*/4$ for all $j \geq j_0$ by requiring j_0 to be larger if necessary.

Given $j \geq j_0$, similarly to the proof of the claim (2.6), there exists a function W satisfying

- W is strictly decreasing on its domain $[a_\infty, b_\infty]$ with $0 < b_\infty - a_\infty \leq y_j - x_j < \epsilon$, is continuous at a_∞ and b_∞ , and is $C^1([a_\infty, b_\infty] \setminus \mathcal{O})$ with \mathcal{O} containing at most countably many points;
- $W(b_\infty) - W(a_\infty) \geq w(s_j, x_j) - w(s_j, y_j) > m^*/4 > 0$;

- $\int_{a_\infty}^{b_\infty} |W'(x)|^2 dx \leq \sum_{i \in \mathbb{N}} \int_{\hat{a}_i}^{\hat{b}_i} |w'(s_j, x)|^2 dx$, where $\bigcup_{i \in \mathbb{N}} (\hat{a}_i, \hat{b}_i) = \{x \in (x_j, y_j) : w'(s_j, x) < 0\}$, with \mathbb{N} being a given set consisting of at most countably many integers, $(\hat{a}_i, \hat{b}_i) \cap (\hat{a}_j, \hat{b}_j) = \emptyset$, $\forall i \neq j$.

As $m'(x) \geq 0$ on $[x_j, y_j]$ and $w'(s_j, x) \leq 0$ on each $[\hat{a}_i, \hat{b}_i]$, the same analysis as in step 3 gives

$$\begin{aligned} 2 \max_{[0,1]} |c(x)| &\geq \int_0^1 \{[w'(s_j, x) - s_j w(s_j, x) m'(x)]^2\} dx \\ &\geq \sum_{i \in \mathbb{N}} \int_{\hat{a}_i}^{\hat{b}_i} |w'(s_j, x)|^2 dx \geq \int_{a_\infty}^{b_\infty} |W'(x)|^2 dx \geq \frac{|W(b_\infty) - W(a_\infty)|^2}{b_\infty - a_\infty} \\ &\geq \frac{|w(s_j, x_j) - w(s_j, y_j)|^2}{y_j - x_j} \geq \frac{(m^*)^2}{16\epsilon}, \end{aligned}$$

which is a contradiction because of the choice of ϵ , and (2.14) is thus verified. So we have proved that $\mu((a_1, a_2)) = 0$. This completes the proof of Lemma 2.6. \square

In fact, we can further show that in Lemma 2.6, $\mu(\{a_1\}) = 0$ when m is strictly increasing and $\mu(\{a_2\}) = 0$ when m is strictly decreasing. In order to cover other general cases, we next consider an interval $[a_1, a_2] \subset [0, 1]$ on which m is non-decreasing and $\{x \in [0, 1] : m'(x) > 0\} \cap (a_1, a_2) \neq \emptyset$. Without loss of generality, we assume that $(a_1, a_1 + \epsilon) \cap \{x \in [0, 1] : m'(x) > 0\} \neq \emptyset$ and $(a_2 - \epsilon, a_2) \cap \{x \in [0, 1] : m'(x) > 0\} \neq \emptyset$, $\forall \epsilon > 0$. Denote by $\bigcup_{l=1}^L (c_l, d_l)$ the set of intervals where m is constant on each $[c_l, d_l] \subset (a_1, a_2)$, and $[c_l, d_l] \cap [c_j, d_j] = \emptyset$, $\forall l \neq j$, where $L \geq 0$ is a finite integer or equal to ∞ . Here, we make a convention that $L = 0$ means that m is strictly increasing on $[a_1, a_2]$. Then we are able to state

Lemma 2.7. $\mu([a_1, a_2] \setminus \bigcup_{l=1}^L (c_l, d_l)) = 0$, and if additionally $0 < a_2 < 1$ and $m'(x) \geq 0$ on $[a_2, a_2 + \epsilon_0]$ for some small $\epsilon_0 > 0$, then $\mu([a_1, a_2] \setminus \bigcup_{l=1}^L (c_l, d_l)) = 0$.

Proof. We first prove

$$(2.15) \quad \mu((a_1, a_2) \setminus \bigcup_{l=1}^L [c_l, d_l]) = 0.$$

When $L < \infty$, the proof is the same as in Lemma 2.6. It remains to consider $L = \infty$. According to our assumption, there exists a sequence $\{z_i\}_{i \geq 1}$ with $a_1 < z_i < a_2$ and $z_i \rightarrow a_2$ as $i \rightarrow \infty$ such that $m'(z_i) > 0$ for each $i \geq 1$. Thus, it is sufficient to show that

$$(2.16) \quad \mu((a_1, z_i) \setminus \bigcup_{l=1}^{\infty} [c_l, d_l]) = 0 \quad \text{for any given } i \geq 1.$$

So from now on, we always fix i .

Note that $\{x \in [0, 1] : m'(x) > 0\} \cap (z_i, a_2) \neq \emptyset$. Then Lemma 2.5 implies that $w(s_j, \hat{x}) \rightarrow 0$ for some $\hat{x} \in (z_i, a_2)$. In view of this fact, one can appeal to the similar argument as in proving (2.6) to conclude that

$$(2.17) \quad \|w(s_j, \cdot)\|_{L^\infty(a_1, z_i)} \leq M \text{ for some positive constant } M, \text{ independent of } j \geq 1.$$

We use the same notation \mathcal{A} , \mathcal{A}_k , $k \geq 1$ as in the proof of Lemma 2.6. Given $\delta = \delta(k)$, (2.12) there remains true. On the other hand, for any given small $\epsilon > 0$, we have

$$(2.18) \quad \sum_{l=N_0+1}^{\infty} |c_l - d_l| < \epsilon \quad \text{for some large } N_0 = N_0(\epsilon).$$

Furthermore, the open set $(a_1, z_i) \setminus \bigcup_{l=1}^{N_0} [c_l, d_l]$ consists of $N^* = N_0 + 1$ open intervals, denoted by $\bigcup_{l=1}^{N^*} (e_l, f_l)$, with each $(e_l, f_l) \subset (a_1, z_i)$ and $[e_l, f_l] \cap [e_j, f_j] = \emptyset$, $\forall l \neq j$. Clearly, $\mathcal{A} \cap (f_l - \rho, f_l) \neq \emptyset$, $\forall 1 \leq l \leq N^*$ for any small $\rho > 0$.

For simplicity, denote

$$\mathcal{B}(\epsilon) = (a_1, z_i) \setminus \left(\mathcal{A} \cup \bigcup_{l=1}^{N_0(\epsilon)} [c_l, d_l] \right).$$

For the above ϵ , we then claim that, given $1 \leq l \leq N^*$, there is a large $k^l = k^l(\epsilon)$, such that

$$(2.19) \quad d(x, \mathcal{A}_k \cap (x, f_l)) \leq \left(1 + \frac{1}{N^*}\right)\epsilon, \quad \forall x \in \mathcal{B} \cap (e_l, f_l - \epsilon/N^*), \quad \text{for all } k \geq k^l.$$

On the contrary, suppose that (2.19) is invalid. Then for some $1 \leq l_0 \leq N^*$, there is a point sequence $x_{k_\eta} \in \mathcal{B} \cap (e_{l_0}, f_{l_0} - \epsilon/N^*)$, such that

$$d(x_{k_\eta}, \mathcal{A}_{k_\eta} \cap (x_{k_\eta}, f_{l_0})) > \left(1 + \frac{1}{N^*}\right)\epsilon, \quad \text{for all } \eta \geq 1.$$

We may assume that $x_{k_\eta} \rightarrow x^* \in \overline{\mathcal{B}} \cap [e_{l_0}, f_{l_0} - \epsilon/N^*]$. As in the proof of Lemma 2.6, we have by sending $\eta \rightarrow \infty$ that

$$d(x^*, \mathcal{A} \cap (x^*, f_{l_0})) \geq \left(1 + \frac{1}{N^*}\right)\epsilon.$$

If $d(x^*, \mathcal{A} \cap (x^*, f_{l_0})) = \infty$, obviously m is constant on $[x^*, f_{l_0}]$, contradicting against the fact that $\mathcal{A} \cap (f_l - \rho, f_l) \neq \emptyset$, $\forall 1 \leq l \leq N^*$ for any small $\rho > 0$. If $(1 + \frac{1}{N^*})\epsilon \leq d(x^*, \mathcal{A} \cap (x^*, f_{l_0})) < \infty$, then it is necessary that m is constant on $(x^*, x^* + (1 + \frac{1}{N^*})\epsilon) \subset [e_{l_0}, f_{l_0}]$. This is also impossible because (2.18) and the definition of $[e_{l_0}, f_{l_0}]$ already imply that $f_{l_0} - e_{l_0} < \epsilon$.

Next we are going to show

$$(2.20) \quad \limsup_{\epsilon \rightarrow 0} \limsup_{j \rightarrow \infty} \left(\epsilon^{-\frac{1}{3}} \|w(s_j, \cdot)\|_{L^\infty(\mathcal{B} \cap (\bigcup_{l=1}^{N^*} (e_l, f_l - \epsilon/N^*)))} \right) < 2.$$

Otherwise, by means of (2.19) and (2.12), we can find a sequence ϵ_{j_κ} satisfying $\epsilon_{j_\kappa} \rightarrow 0$ as $\kappa \rightarrow \infty$ and two points sequences $x_{j_\kappa} \in \mathcal{B}(\epsilon_{j_\kappa}) \cap (\bigcup_{l=1}^{N^*} (e_l(\epsilon_{j_\kappa}), f_l(\epsilon_{j_\kappa}) - \epsilon_{j_\kappa}/N^*(\epsilon_{j_\kappa})))$ and $y_{j_\kappa} \in \mathcal{A}$ satisfying $0 < y_{j_\kappa} - x_{j_\kappa} < (1 + \frac{1}{N^*(\epsilon_{j_\kappa})})\epsilon_{j_\kappa} \leq 2\epsilon_{j_\kappa}$ such that $w(s_{j_\kappa}, x_{j_\kappa}) > (\epsilon_{j_\kappa})^{1/3}$ and $w(s_{j_\kappa}, y_{j_\kappa}) < \epsilon_{j_\kappa}$, $\forall \kappa \geq 1$.

Given large κ , by the argument similar to that of deriving the claim (2.6), we can find a function W such that

- W is strictly decreasing on its domain $[a_\infty, b_\infty]$ with $0 < b_\infty - a_\infty \leq y_{j_\kappa} - x_{j_\kappa} \leq 2\epsilon_{j_\kappa}$, is continuous at a_∞ and b_∞ , and is $C^1([a_\infty, b_\infty] \setminus \mathcal{O})$ with \mathcal{O} containing at most countably many points;
- $W(b_\infty) - W(a_\infty) \geq w(s_{j_\kappa}, x_{j_\kappa}) - w(s_{j_\kappa}, y_{j_\kappa}) > (\epsilon_{j_\kappa})^{1/3} - \epsilon_{j_\kappa} > 0$;

- $\int_{a_\infty}^{b_\infty} |W'(x)|^2 dx \leq \sum_{i \in \mathbb{N}} \int_{\hat{a}_i}^{\hat{b}_i} |w'(s_j, x)|^2 dx$, where $\bigcup_{i \in \mathbb{N}} (\hat{a}_i, \hat{b}_i) = \{x \in (x_{j_\kappa}, y_{j_\kappa}) : w'(s_{j_\kappa}, x) < 0\}$, with \mathbb{N} being a given set consisting of at most countably many integers, $(\hat{a}_i, \hat{b}_i) \cap (\hat{a}_j, \hat{b}_j) = \emptyset$, $\forall i \neq j$.

Using $m'(x) \geq 0$ on $[x_{j_\kappa}, y_{j_\kappa}]$ and $w'(s_{j_\kappa}, x) \leq 0$ on each $[\hat{a}_i, \hat{b}_i]$, the same analysis as in step 3 of the proof of Lemma 2.6 deduces

$$2 \max_{[0,1]} |c(x)| \geq \frac{|w(s_{j_\kappa}, y_{j_\kappa}) - w(s_{j_\kappa}, x_{j_\kappa})|^2}{y_{j_\kappa} - x_{j_\kappa}} \geq \frac{[(\epsilon_{j_\kappa})^{1/3} - \epsilon_{j_\kappa}]^2}{2\epsilon_{j_\kappa}} \rightarrow \infty,$$

as $\kappa \rightarrow \infty$. This contradiction yields (2.20).

We recall that Lemma 2.5 implies $\mu(\mathcal{A}_k) = 0, \forall k \geq 1$ and therefore $\mu(\mathcal{A}) = 0$. Combing this fact, (2.17), (2.18) and (2.20), from (1.10) it is not hard to see that

$$\begin{aligned} \mu((a_1, z_i) \setminus \bigcup_{l=1}^L [c_l, d_l]) &\leq \mu((a_1, z_i) \setminus \bigcup_{l=1}^{N_0} [c_l, d_l]) = \mu(\mathcal{B} \cap \bigcup_{l=1}^{N^*} (e_l, f_l)) \\ &= \mu(\mathcal{B} \cap \bigcup_{l=1}^{N^*} (e_l, f_l - \epsilon/N^*)) + \mu(\mathcal{B} \cap \bigcup_{l=1}^{N^*} [f_l - \epsilon/N^*, f_l)) \\ &\leq \sum_{l=1}^{N^*} \mu(\mathcal{B} \cap (e_l, f_l - \epsilon/N^*)) + \sum_{l=1}^{N^*} \mu([f_l - \epsilon/N^*, f_l)) \\ &\leq 4\epsilon^{2/3} + 2M^2\epsilon. \end{aligned}$$

As ϵ can be arbitrarily small, we get (2.16), and so (2.15) holds.

In what follows, we will show $\mu(\{a_1\}) = 0$. Assume that $a_1 \in (0, 1)$. Since m is non-decreasing on $[a_1, a_2]$ and $(a_1, a_1 + \epsilon) \cap \{x \in [0, 1] : m'(x) > 0\} \neq \emptyset, \forall \epsilon > 0$, we know from (1.8) that there are two cases to occur: Case (i) $m'(x) \geq 0$ on $[a_1 - \delta_0, a_1]$ for some small δ_0 ; Case (ii) $m'(x) \leq 0$ on $[a_1 - \delta_0, a_1]$ for some small δ_0 and $\{m'(x) < 0\} \cap (a_1 - \epsilon, a_1) \neq \emptyset$ for any small ϵ .

In each case, by the same argument as obtaining the claim (2.6) in the proof of Lemma 2.6 we have

$$(2.21) \quad \limsup_{j \rightarrow \infty} \|w(s_j, \cdot)\|_{L^\infty((a_1 - \delta_0, a_1 + \delta_0))} = m_* < \infty.$$

As a consequence, for any small $0 < \epsilon < \delta_0$, by taking $\zeta = 1$ on $[a_1 - \epsilon, a_1 + \epsilon]$, $\zeta = 0$ on $[0, a_1 - 2\epsilon] \cup [a_1 + 2\epsilon, 1]$, and $0 \leq \zeta \leq 1$ otherwise so that $\zeta \in C([0, 1])$, from (1.10) it easily follows that

$$\mu(\{a_1\}) \leq \mu([a_1 - \epsilon, a_1 + \epsilon]) \leq \lim_{j \rightarrow \infty} \int_0^1 w^2(s_j, x) \zeta(x) dx \leq 4m_*^2 \epsilon,$$

which implies $\mu(\{a_1\}) = 0$.

Clearly the above analysis can be used to handle the case $a_1 = 0$ and the end points c_l, d_l of the interval $[c_l, d_l]$ for each $l \geq 1$, and obtain $\mu(\{c_l\}) = \mu(\{d_l\}) = 0$. There are at most countably many such end points. Thus, the countable additivity of the Radon measure ensures $\mu([a_1, a_2] \setminus \bigcup_{l=1}^L (c_l, d_l)) = 0$. If additionally $0 < a_2 < 1$ and $m'(x) \geq 0$ on $[a_2, a_2 + \epsilon_0]$ for some small $\epsilon_0 > 0$, by the analysis as above we have $\mu(\{a_2\}) = 0$. The proof of Lemma 2.7 is ended. \square

Remark 2.1. *The parallel assertion to Lemma 2.7 holds: Assume that m is non-increasing on $[a_1, a_2] \subset [0, 1]$, $(a_1, a_1 + \epsilon) \cap \{x \in [0, 1] : m'(x) < 0\} \neq \emptyset$ and $(a_2 - \epsilon, a_2) \cap \{x \in [0, 1] : m'(x) < 0\} \neq \emptyset$, $\forall \epsilon > 0$. Denote by $\bigcup_{l=1}^L (c_l, d_l)$ the set of all the intervals that m is constant on each $[c_l, d_l] \subset (a_1, a_2)$, and $[c_l, d_l] \cap [c_j, d_j] = \emptyset$, $\forall l \neq j$, where $L \geq 0$ is a finite integer or equal to ∞ . Then we have $\mu((a_1, a_2) \setminus \bigcup_{l=1}^L (c_l, d_l)) = 0$, and if additionally $0 < a_1 < 1$ and $m'(x) \leq 0$ on $[a_1 - \epsilon_0, a_1]$ for some small $\epsilon_0 > 0$, then $\mu([a_1, a_2] \setminus \bigcup_{l=1}^L (c_l, d_l)) = 0$.*

With the above preparation, we are now in a position to prove Theorem 1.2 in the case of Neumann boundary condition. In order to avoid using complicated notation as well as tedious analysis, we consider two special functions m which carry typical kinds of degeneracy, and derive the limit $\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{N}}(s)$ for each such chosen m . Then, it will be not hard to see that Theorem 1.2 follows by using the similar argument as for those two cases. Keeping such a strategy in mind, we first choose a function m such that

(T1): m is strictly increasing on $[0, a_1] \cup [a_2, a_3] \cup [a_5, 1]$, and is strictly decreasing on $[a_1, a_2] \cup [a_4, a_5]$, and m is constant on $[a_3, a_4]$, for positive constants $0 < a_1 < a_2 < a_3 < a_4 < a_5 < 1$. See Figure 7. For such given m , we can conclude that

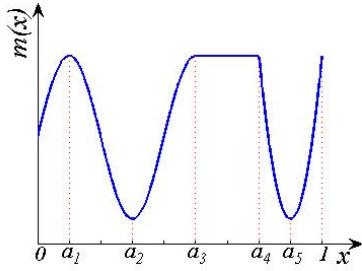


Figure 7

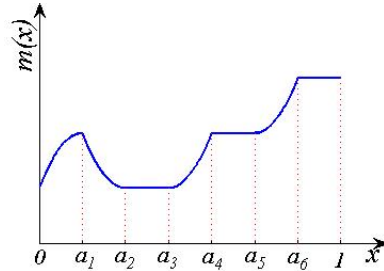


Figure 8

Theorem 2.1. *Assume that (T1) holds, then*

$$\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{N}}(s) = \min\{c(a_1), c(1), \lambda_1^{\mathcal{NN}}(a_3, a_4)\}.$$

Proof. By Lemma 2.1, we have

$$(2.22) \quad \limsup_{s \rightarrow \infty} \lambda_1^{\mathcal{N}}(s) = \lambda^* \leq \min\{c(a_1), c(1), \lambda_1^{\mathcal{NN}}(a_3, a_4)\}.$$

So it suffices to prove

$$(2.23) \quad \liminf_{s \rightarrow \infty} \lambda_1^{\mathcal{N}}(s) = \lambda_* \geq \min\{c(a_1), c(1), \lambda_1^{\mathcal{NN}}(a_3, a_4)\}.$$

First of all, Lemmas 2.6, 2.7, Remark 2.1 and the countable additivity of the probability measure tell us that

$$(2.24) \quad \mu([0, a_1] \cup (a_1, a_3] \cup [a_4, 1)) = 0.$$

Thus, there holds

$$(2.25) \quad \mu(\{a_1\}) + \mu((a_3, a_4)) + \mu(\{1\}) = 1.$$

On the other hand, for any small $\epsilon > 0$, from (1.5) and the assumption that m is constant on $[a_3, a_4]$ it follows that

$$\begin{aligned}
 \lambda_1^{\mathcal{N}}(s_j) &= \int_0^1 \{|w'(s_j, x) - s_j w(s_j, x) m'(x)|^2 + c(x) w^2(s_j, x)\} dx \\
 (2.26) \quad &\geq \left(\int_0^{a_1-\epsilon} + \int_{a_1+\epsilon}^{a_3} + \int_{a_4}^{1-\epsilon} \right) c(x) w^2(s_j, x) dx + \left(\int_{a_1-\epsilon}^{a_1+\epsilon} + \int_{1-\epsilon}^1 \right) c(x) w^2(s_j, x) dx \\
 &\quad + \int_{a_3}^{a_4} \{|w'(s_j, x)|^2 + c(x) w^2(s_j, x)\} dx.
 \end{aligned}$$

Appealing to Lemma 2.2(ii), we get

$$(2.27) \quad \left(\int_0^{a_1-\epsilon} + \int_{a_1+\epsilon}^{a_3} + \int_{a_4}^{1-\epsilon} \right) c(x) w^2(s_j, x) dx \rightarrow 0, \quad \text{as } j \rightarrow \infty,$$

and by Lemma 2.3, we have

$$(2.28) \quad \lim_{j \rightarrow \infty} \left(\int_{a_1-\epsilon}^{a_1+\epsilon} + \int_{1-\epsilon}^1 \right) c(x) w^2(s_j, x) dx \geq \min_{[a_1-\epsilon/2, a_1+\epsilon/2]} c(x) \mu(\{a_1\}) + \min_{[1-\epsilon/2, 1]} c(x) \mu(\{1\}).$$

To handle the last integral in (2.26), let us assume, for the moment, that there is a sequence of $\{w(s_j, \cdot)\}$, still labelled by itself, such that

$$(2.29) \quad \int_{a_3}^{a_4} w^2(s_j, x) dx > 0, \quad \text{for all } j \geq 1,$$

and hence set

$$v(s_j, x) = \frac{w(s_j, x)}{\sqrt{\int_{a_3}^{a_4} w^2(s_j, x) dx}}.$$

Thus, $v(s_j, x)$ satisfies $\int_{a_3}^{a_4} v^2(s_j, x) dx = 1$ for each $j \geq 1$. By the variational characterization of $\lambda_1^{\mathcal{NN}}(a_3, a_4)$, combined with (2.24) and Lemma 2.2(iii), one finds

$$\begin{aligned}
 \int_{a_3}^{a_4} \{|w'(s_j, x)|^2 + c(x) w^2(s_j, x)\} dx &= \int_{a_3}^{a_4} w^2(s_j, x) dx \int_{a_3}^{a_4} \{|v'(s_j, x)|^2 + c(x) v^2(s_j, x)\} dx \\
 (2.30) \quad &\geq \lambda_1^{\mathcal{NN}}(a_3, a_4) \int_{a_3}^{a_4} w^2(s_j, x) dx \\
 &\rightarrow \lambda_1^{\mathcal{NN}}(a_3, a_4) \mu((a_3, a_4)), \quad \text{as } j \rightarrow \infty.
 \end{aligned}$$

By means of (2.26), (2.27), (2.28) and (2.30), we deduce

$$\lambda_* = \liminf_{j \rightarrow \infty} \lambda_1^{\mathcal{N}}(s_j) \geq \min_{[a_1-\epsilon/2, a_1+\epsilon/2]} c(x) \mu(\{a_1\}) + \min_{[1-\epsilon/2, 1]} c(x) \mu(\{1\}) + \lambda_1^{\mathcal{NN}}(a_3, a_4) \mu((a_3, a_4)).$$

Since $\epsilon > 0$ is arbitrary, by sending $\epsilon \rightarrow 0$ in the above inequality and using (2.25), we obtain

$$\begin{aligned}
 \lambda_* &\geq c(a_1) \mu(\{a_1\}) + c(1) \mu(\{1\}) + \lambda_1^{\mathcal{NN}}(a_3, a_4) \mu((a_3, a_4)) \\
 (2.31) \quad &\geq \min\{c(a_1), c(1), \lambda_1^{\mathcal{NN}}(a_3, a_4)\}.
 \end{aligned}$$

If (2.29) does not hold, then $\int_{a_3}^{a_4} w^2(s_j, x) dx = 0$ for all large j , and in turn $w(s_j, \cdot) \equiv 0$ on $[a_3, a_4]$ since $w(s_j, \cdot) \in C([a_3, a_4])$. Hence, the last integral in (2.26) converges to zero. In

addition, Lemma 2.2(i) asserts $\mu([a_3, a_4]) = 0$, and so $\mu(\{a_1\}) + \mu(\{1\}) = 1$ due to (2.25). Using these facts, (2.27) and (2.28), it follows from (2.26) and (2.22) that

$$\min\{c(a_1), c(1), \lambda_1^{\mathcal{NN}}(a_3, a_4)\} \geq \lambda^* \geq \lambda_* \geq c(a_1)\mu(\{a_1\}) + c(1)\mu(\{1\}) \geq \min\{c(a_1), c(1)\},$$

which thereby implies $\lambda_1^{\mathcal{NN}}(a_3, a_4) \geq \min\{c(a_1), c(1)\}$. As a result, (2.31) remains true and (2.23) is established. This completes the proof. \square

Remark 2.2. A direct check of the proof of Theorem 2.1 shows that the support of the probability measure μ is the set where the limit $\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{N}}(s)$ is attained; for example, if $\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{N}}(s) = c(a_1) < \min\{c(1), \lambda_1^{\mathcal{NN}}(a_3, a_4)\}$, then $\mu(\{a_1\}) = 1$ and $\mu([0, a_1] \cup (a_1, 1]) = 0$, and if $\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{N}}(s) = c(a_1) = c(1) < \lambda_1^{\mathcal{NN}}(a_3, a_4)$, then $\mu(\{a_1, 1\}) = 1$ and $\mu([0, a_1] \cup (a_1, 1)) = 0$. Conversely, the limit is attained on the support of μ ; for example, if $\mu(\{a_1\}) > 0$, then $\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{N}}(s) = c(a_1)$ (and so $c(a_1) \leq \min\{c(1), \lambda_1^{\mathcal{NN}}(a_3, a_4)\}$), and if $\mu((a_3, a_4)) > 0$, then $\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{N}}(s) = \lambda_1^{\mathcal{NN}}(a_3, a_4)$ (and so $\lambda_1^{\mathcal{NN}}(a_3, a_4) \leq \min\{c(a_1), c(1)\}$). A similar comment also applies to Theorem 2.2 below.

We next choose another typical function m satisfying

(T2): m is strictly increasing on $[0, a_1] \cup [a_3, a_4] \cup [a_5, a_6]$, and is strictly decreasing on $[a_1, a_2]$, and m is constant on $[a_2, a_3]$, $[a_4, a_5]$ and $[a_6, 1]$, for positive constants $0 < a_1 < a_2 < a_3 < a_4 < a_5 < a_6 < 1$. See Figure 8. For such a given function m , we have

Theorem 2.2. Assume that (T2) holds, then

$$\lim_{s \rightarrow \infty} \lambda_1^{\mathcal{N}}(s) = \min\{c(a_1), \lambda_1^{\mathcal{DD}}(a_2, a_3), \lambda_1^{\mathcal{ND}}(a_4, a_5), \lambda_1^{\mathcal{NN}}(a_6, 1)\}.$$

Proof. By Lemma 2.1, we have

$$(2.32) \quad \lambda^* \leq \min\{c(a_1), \lambda_1^{\mathcal{DD}}(a_2, a_3), \lambda_1^{\mathcal{ND}}(a_4, a_5), \lambda_1^{\mathcal{NN}}(a_6, 1)\}.$$

In addition, applying Lemmas 2.6, 2.7 and Remark 2.1, we obtain

$$(2.33) \quad \mu([0, a_1] \cup (a_1, a_2] \cup [a_3, a_4] \cup [a_5, a_6]) = 0$$

and so

$$(2.34) \quad \mu(\{a_1\}) + \mu((a_2, a_3)) + \mu((a_4, a_5)) + \mu(a_6, 1] = 1.$$

In what follows we are going to prove

$$(2.35) \quad \lambda_* \geq \min\{c(a_1), \lambda_1^{\mathcal{DD}}(a_2, a_3), \lambda_1^{\mathcal{ND}}(a_4, a_5), \lambda_1^{\mathcal{NN}}(a_6, 1)\}.$$

To achieve the aim, we first need the following fact: for a generic sequence $\{s_j\}$, there holds

$$(2.36) \quad w(s_j, x) \rightarrow w_*(x) \text{ uniformly on } [a_2, a_3], \text{ as } j \rightarrow \infty,$$

for some $w_* \in C([a_2, a_3])$. Indeed, since $m(x)$ is constant on $[a_2, a_3]$, we get

$$\begin{aligned} \max_{[0,1]} |c(x)| &\geq \lambda_1^{\mathcal{N}}(s_j) = \int_0^1 [|w'(s_j, x) - s_j w(s_j, x) m'(x)|^2 + c(x) w^2(s_j, x)] dx \\ &\geq \int_{a_2}^{a_3} |w'(s_j, x)|^2 dx - \max_{[0,1]} |c(x)|, \end{aligned}$$

which in turn gives

$$\int_{a_2}^{a_3} w^2(s_j, x) dx + \int_{a_2}^{a_3} |w'(s_j, x)|^2 dx \leq 1 + 2 \max_{[0,1]} |c(x)|.$$

Notice that $H^1((a_2, a_3))$ is compactly embedded into $C([a_2, a_3])$. Thus, (2.36) holds true.

As m is constant on $[a_2, a_3]$ and $w(s_j, \cdot)$ satisfies

$$(2.37) \quad -w''(s_j, x) + c(x)w(s_j, x) = \lambda_1^N(s_j)w(s_j, x), \quad a_2 < x < a_3.$$

By a standard compactness consideration, for a subsequence of $\{s_j\}$, denoted by itself for simplicity, satisfying $\lambda_1^N(s_j) \rightarrow \lambda_*$ as $j \rightarrow \infty$, it is easily seen that $w(s_j, x) \rightarrow w_*(x)$ in $C_{loc}^1((a_2, a_3))$. Combining this fact and (2.36), one can use (2.37) to further conclude that $w(s_j, x) \rightarrow w_*(x)$ in $C^1([a_2, a_3])$. So $w_* \in C^1([a_2, a_3])$ solves in the weak sense

$$-w_*''(x) + c(x)w_*(x) = \lambda_* w_*(x), \quad a_2 < x < a_3.$$

In the sequel, we will show $w_*(a_2) = w_*(a_3) = 0$. We argue indirectly and suppose that $w_*(a_2) > 0$. Since $w(s_j, a_2) \rightarrow w_*(a_2) > 0$, $w(s_j, a_2) \geq \frac{1}{2}w_*(a_2)$ for all large j . On the other hand, thanks to $\{m'(x) < 0\} \cap (a_2 - \rho, a_2) \neq \emptyset, \forall \rho > 0$, for any given small constant δ with $0 < \delta < \frac{1}{4}w_*(a_2)$, there exist two sequences $\{j_k\}_{k=1}^\infty$ and $\{\rho_k\}_{k=1}^\infty$ satisfying $j_k \rightarrow \infty, \rho_k \rightarrow 0$ as $k \rightarrow \infty$, such that $w(s_{j_k}, a_2 - \rho_k) \leq \delta, \forall k \geq 1$. Noticing that $\rho_k \rightarrow 0$ as $k \rightarrow \infty$, and

$$w(s_{j_k}, a_2) \geq \frac{1}{2}w_*(a_2), \quad w(s_{j_k}, a_2 - \rho_k) \leq \delta < \frac{1}{4}w_*(a_2),$$

one can use the similar analysis to that in the proof of Lemma 2.6 to arrive at a contradiction. Thus $w_*(a_2) = 0$ holds. Similarly, we have $w_*(a_3) = 0$.

To summarize, the above argument asserts that $w(s_j, x) \rightarrow w_*(x)$ in $C^1([a_2, a_3])$ for some function w_* , where $w_* \geq 0$ solves

$$(2.38) \quad -w_*''(x) + c(x)w_*(x) = \lambda_* w_*(x), \quad a_2 < x < a_3; \quad w_*(a_2) = w_*(a_3) = 0.$$

Similarly, on $[a_4, a_5]$ and $[a_6, 1]$, by passing to a further subsequence if necessary, we know that $w(s_j, x) \rightarrow w_*(x)$ in $C^1([a_4, a_5])$, and $w_* \geq 0$ in $[a_4, a_5]$ and solves

$$(2.39) \quad -w_*''(x) + c(x)w_*(x) = \lambda_* w_*(x), \quad a_4 < x < a_5; \quad w_*(a_5) = 0,$$

and $w(s_j, x) \rightarrow w_*(x)$ in $C^1([a_6, 1])$, and $w_* \geq 0$ in $[a_6, 1]$ and solves

$$(2.40) \quad -w_*''(x) + c(x)w_*(x) = \lambda_* w_*(x), \quad a_6 < x < 1.$$

Now, given any small $\epsilon > 0$, by means of (1.5) and the assumption on m , we obtain

$$(2.41) \quad \begin{aligned} \lambda_1^N(s_j) &= \int_0^1 \{|w'(s_j, x) - s_j w(s_j, x)m'(x)|^2 + c(x)w^2(s_j, x)\} dx \\ &\geq \left(\int_0^{a_1-\epsilon} + \int_{a_1+\epsilon}^{a_2} + \int_{a_3}^{a_4} + \int_{a_5}^{a_6} \right) c(x)w^2(s_j, x) dx + \int_{a_1-\epsilon}^{a_1+\epsilon} c(x)w^2(s_j, x) dx \\ &\quad + \left(\int_{a_2}^{a_3} + \int_{a_4}^{a_5} + \int_{a_6}^1 \right) \{|w'(s_j, x)|^2 + c(x)w^2(s_j, x)\} dx. \end{aligned}$$

Using Lemma 2.2(ii), it is easily seen that the first term in (2.41) converges to zero as $j \rightarrow \infty$. By virtue of Lemma 2.3, one has

$$(2.42) \quad \lim_{j \rightarrow \infty} \int_{a_1 - \epsilon}^{a_1 + \epsilon} c(x) w^2(s_j, x) dx \geq \min_{[a_1 - \epsilon/2, a_1 + \epsilon/2]} c(x) \mu(\{a_1\}).$$

Assume that there is a subsequence of $\{s_j\}$, labelled by itself again, such that

$$(2.43) \quad \int_{a_6}^1 w^2(s_j, x) dx > 0 \quad \text{for all large } j,$$

and

$$(2.44) \quad w_*(x) \geq, \neq 0 \quad \text{on } [a_2, a_3], \quad w_*(x) \geq, \neq 0 \quad \text{on } [a_4, a_5].$$

Let us denote

$$v_1(s_j, x) = \frac{w(s_j, x)}{\sqrt{\int_{a_2}^{a_3} w^2(s_j, x) dx}}, \quad v_2(s_j, x) = \frac{w(s_j, x)}{\sqrt{\int_{a_4}^{a_5} w^2(s_j, x) dx}}, \quad v_3(s_j, x) = \frac{w(s_j, x)}{\sqrt{\int_{a_6}^1 w^2(s_j, x) dx}}.$$

Thus, we have

$$v_1 \rightarrow \frac{w_*}{\sqrt{\int_{a_2}^{a_3} w_*^2(x) dx}} =: w_1 \text{ in } C^1([a_2, a_3]), \quad v_2 \rightarrow \frac{w_*}{\sqrt{\int_{a_4}^{a_5} w_*^2(x) dx}} =: w_2 \text{ in } C^1([a_4, a_5]),$$

and $\int_{a_6}^1 v_3^2(s_j, x) dx = 1$ for each $j \geq 1$, where $w_1, w_2 \geq, \neq 0$ with $w_1(a_2) = w_1(a_3) = w_2(a_5) = 0$ and $\int_{a_2}^{a_3} w_1^2(x) dx = \int_{a_4}^{a_5} w_2^2(x) dx = 1$.

In view of the variational characterizations for $\lambda_1^{\mathcal{DD}}(a_2, a_3)$, $\lambda_1^{\mathcal{ND}}(a_4, a_5)$, $\lambda_1^{\mathcal{NN}}(a_6, 1)$, similarly to the proof of Theorem 2.1 we have, as $j \rightarrow \infty$,

$$(2.45) \quad \begin{aligned} \int_{a_2}^{a_3} \{|w'(s_j, x)|^2 + c(x) w^2(s_j, x)\} dx &= \int_{a_2}^{a_3} w^2(s_j, x) dx \int_{a_2}^{a_3} \{|v_1'(s_j, x)|^2 + c(x) v_1^2(s_j, x)\} dx \\ &\rightarrow \int_{a_2}^{a_3} w_*^2(x) dx \int_{a_2}^{a_3} \{|w_1'(x)|^2 + c(x) w_1^2(x)\} dx \\ &\geq \lambda_1^{\mathcal{DD}}(a_2, a_3) \mu((a_3, a_4)), \end{aligned}$$

and

$$(2.46) \quad \int_{a_4}^{a_5} \{|w'(s_j, x)|^2 + c(x) w^2(s_j, x)\} dx \geq \lambda_1^{\mathcal{ND}}(a_4, a_5) \mu((a_4, a_5)),$$

and

$$(2.47) \quad \int_{a_6}^1 \{|w'(s_j, x)|^2 + c(x) w^2(s_j, x)\} dx \geq \lambda_1^{\mathcal{NN}}(a_6, 1) \mu((a_6, 1]).$$

Making use of (2.34), (2.41), (2.42), (2.45), (2.46) and (2.47), we can easily deduce (2.35) when (2.44) and (2.43) hold.

If one of (2.44) and (2.43) is not satisfied, the analysis of Theorem 2.1 can be adapted to obtain (2.35). So the desired limit is derived. \square

Following the similar argument to that of Theorems 2.1 and 2.2, taking into account Lemma 2.7 and its proof, one can obtain Theorem 1.2 for problem (1.2).

2.2. The eigenvalue problems (1.1) and (1.6). In the previous subsection, we have proved Theorem 1.2 in the Neumann boundary condition case (that is, $\ell_1 = \ell_2 = 0$). This subsection concerns problem (1.1) in the case of $\ell_1 + \ell_2 > 0$ and the periodic problem (1.6). We first give

Proof of Theorem 1.2 for $\ell_1 + \ell_2 > 0$ We first consider the case of $\hbar_i, \ell_i > 0$ ($i = 1, 2$). Then through the same transformation $w = e^{sm}\varphi$ as for the Neumann problem, (1.1) is equivalent to

$$(2.48) \quad \begin{cases} -w''(x) + [s^2(m'(x))^2 + sm''(x) + c(x)]w(x) = \lambda_1(s)w(x), & 0 < x < 1, \\ (s\hbar_1 + \ell_1)w(0) - \hbar_1 w'(0) = (-s\hbar_2 + \ell_2)w(1) + \hbar_2 w'(1) = 0. \end{cases}$$

For each $s \in \mathbb{R}$, $\lambda_1(s)$ enjoys the variational characterization

$$\begin{aligned} \lambda_1(s) &= \min_{\int_0^1 e^{2sm}\varphi^2 dx = 1} \left\{ \int_0^1 e^{2sm}[(\varphi')^2 + c\varphi^2]dx + \frac{\ell_1}{\hbar_1} e^{2sm(0)}\varphi^2(0) + \frac{\ell_2}{\hbar_2} e^{2sm(1)}\varphi^2(1) \right\} \\ &= \min_{\int_0^1 w^2 dx = 1} \left\{ \int_0^1 [(w' - swm')^2 + cw^2]dx + \frac{\ell_1}{\hbar_1} w^2(0) + \frac{\ell_2}{\hbar_2} w^2(1) \right\}. \end{aligned}$$

Additionally, $\min_{x \in [0,1]} c(x) \leq \lambda_1(s)$, $\forall s \in \mathbb{R}$.

As in the Neumann case, we use $w(s, \cdot)$ with $\int_0^1 w^2(s, x)dx = 1$ to denote the principal eigenfunction of (2.48), and by the weak compactness of $\{w^2(s, \cdot)\}_{s>0}$, for a generic sequence $\{s_j\}_{j=1}^\infty$ satisfying $s_j \rightarrow \infty$ as $j \rightarrow \infty$, we assume that

$$\lim_{j \rightarrow \infty} \int_0^1 w^2(s_j, x)\zeta(x)dx = \int_{[0,1]} \zeta(x)\mu(dx), \quad \forall \zeta \in C([0, 1])$$

for a unique probability measure μ .

We assume that m has at least one interior point of local maximum. Clearly, under our hypothesis (1.8), m contains at least one isolated interior point or segment of local maximum. Hence, one can easily modify the argument of Lemma 2.1 to show that $\limsup_{s \rightarrow \infty} \lambda_1(s) < \infty$.

Conversely, if m has no interior point of local maximum, then under the assumption (1.8), there are only three possibilities: 0 is the unique isolated point of local maximum, and so m must be strictly decreasing on $[0, 1]$; 1 is the unique isolated point of local maximum, and so m must be strictly increasing on $[0, 1]$; only 0 and 1 are the isolated points of local maximum, and so m must be strictly decreasing on $[0, x_0]$ while strictly decreasing on $[x_0, 1]$ for some $x_0 \in (0, 1)$. We only consider the first case and the other two cases can be tackled similarly. In order to show $\lim_{s \rightarrow \infty} \lambda_1(s) = \infty$, we proceed indirectly and suppose that $\liminf_{s \rightarrow \infty} \lambda_1(s) = \lambda_* \in [\min_{x \in [0,1]} c(x), \infty)$. Thus, $\lim_{j \rightarrow \infty} \lambda_1(s_j) = \lambda_*$ for some sequence $\{s_j\}_{j \geq 1}$ satisfying $s_j \rightarrow \infty$ as $j \rightarrow \infty$. Then Lemma 2.7 gives $\mu((0, 1]) = 0$, and in turn $\mu(\{0\}) = 1$. By virtue of $\int_0^1 w^2(s_i, x)dx = 1$ for each $j \geq 1$, with the help of Lemma 2.2(ii), one can further assert that

$$\limsup_{j \rightarrow \infty} \|w(s_j, \cdot)\|_{L^\infty((0,1))} = \infty.$$

Hence, the analysis similar to that of Lemma 2.6 results in a contradiction, and so it is necessary that $\lim_{s \rightarrow \infty} \lambda_1(s) = \infty$.

In summary, the above analysis shows that when $\hbar_i, \ell_i > 0$ ($i = 1, 2$), $\lim_{s \rightarrow \infty} \lambda_1(s) = \infty$ if and only if $\mathcal{M} = \mathcal{M}_1 \subset \{0, 1\}$. Moreover, if $\lim_{s \rightarrow \infty} \lambda_1(s) < \infty$, we first use the same argument as in

Lemma 2.1 to deduce $\limsup_{s \rightarrow \infty} \lambda_1(s) \leq \hat{\lambda}$, where $\hat{\lambda}$ is the limiting value given in Theorem 1.2(ii). So as above, we can claim that $\mu(\{0, 1\} \cap \mathcal{M}_1) = 0$. Combined with this fact, the analysis of Theorems 2.1 and 2.2 can be easily adapted to prove Theorem 1.2(ii); the details are omitted here to avoid unnecessary repetition. Here the only point we want to stress is that when one of $\mathcal{M}_i (i = 6, 7, 8, 9)$ is nonempty, for instance, if $[0, a^I] \in \mathcal{M}_6$, since m is constant on $[0, a^I]$, through the transformation $w = e^{sm}\varphi$, we notice that w satisfies the same boundary condition as φ at the endpoint 0: $-\hbar_1 w'(0) + \ell_1 w(0) = 0$.

When $\hbar_1 = \hbar_2 = 0$ (so $\ell_1, \ell_2 > 0$), (1.2) becomes Dirichlet eigenvalue problem (1.7). Thus, we have

$$\begin{aligned} \lambda_1(s) &= \min_{\varphi \in H_0^1, \int_0^1 e^{2sm} \varphi^2 dx = 1} \int_0^1 e^{2sm} [(\varphi')^2 + c\varphi^2] dx \\ &= \min_{w \in H_0^1, \int_0^1 w^2 dx = 1} \int_0^1 [(w' - swm')^2 + cw^2] dx, \end{aligned}$$

and when $\hbar_1 = 0, \hbar_2, \ell_2 > 0$, we have

$$\begin{aligned} \lambda_1(s) &= \min_{\varphi \in H_*^1, \int_0^1 e^{2sm} \varphi^2 dx = 1} \int_0^1 e^{2sm} [(\varphi')^2 + c\varphi^2] dx + \frac{\ell_2}{\hbar_2} \varphi^2(1) \\ &= \min_{w \in H_*^1, \int_0^1 w^2 dx = 1} \int_0^1 [(w' - swm')^2 + cw^2] dx + \frac{\ell_2}{\hbar_2} w^2(1), \end{aligned}$$

where $H_*^1 = \{g \in H^1(0, 1) : g(0) = 0\}$. By such variational characterizations, we can use the argument similar to the above to obtain the desired results. The remaining cases can be handled similarly, and the detailed are omitted. This completes the proof of Theorem 1.2. \square

At last, we give the

Proof of Theorem 1.3 Recall that the principal eigenvalue $\lambda_1^{\mathcal{P}}(s)$ of (1.6) can be variationally characterized as

$$\begin{aligned} \lambda_1^{\mathcal{P}}(s) &= \min_{\varphi \text{ is } 1\text{-periodic}, \int_0^1 e^{2sm} \varphi^2 dx = 1} \int_0^1 e^{2sm} [(\varphi')^2 + c\varphi^2] dx \\ &= \min_{w \text{ is } 1\text{-periodic}, \int_0^1 w^2 dx = 1} \int_0^1 [(w' - swm')^2 + cw^2] dx. \end{aligned}$$

As before, making use of the transformation $w = e^{sm}\varphi$, we obtain from (1.6) that

$$\begin{cases} -w''(x) + [s^2(m'(x))^2 + sm''(x) + c(x)]w(x) = \lambda_1^{\mathcal{P}}(s)w(x), & x \in \mathbb{R}, \\ w(x) = w(x+1), & x \in \mathbb{R}. \end{cases}$$

Consequently, Theorem 1.3 follows from the same analysis as that of Theorem 1.2. \square

3. THE PRINCIPAL EIGENFUNCTION: PROOF OF THEOREM 1.4

This section is devoted to the study of the asymptotic behavior of the principal eigenfunction of (1.4) as $s \rightarrow \infty$. It is easily seen from the analysis in the previous section that the assertions (2)-(9) of Theorem 1.4 hold. Hence, it remains to prove the assertion (1) of Theorem 1.4. To the end, we use the techniques introduced in [3] with necessary modifications.

We first consider the case $x_0 \in (0, 1)$. By our assumption, there exist small positive constants a and R , and a large positive constant A , such that

$$(3.1) \quad |m'(x)|^2 \geq a|x - x_0|^{2(k^*-1)}, \quad \forall x \in B(x_0, 4R) \subset (0, 1),$$

and

$$(3.2) \quad |m''(x)| \leq A|x - x_0|^{k^*-2}, \quad \forall x \in B(x_0, 4R) \subset (0, 1).$$

As in [3], through scaling, without loss of generality we may assume that

$$(3.3) \quad \|m'(x)\|_{L^\infty(0,1)} + \|m''(x)\|_{L^\infty(0,1)} + \max_{[0,1]} c(x) - \min_{[0,1]} c(x) \leq \frac{1}{2},$$

and define for convenience

$$q(s, x) = s^2|m'(x)|^2 + sm''(x) + c(x) - \lambda_1^{\mathcal{N}}(s).$$

Then, we have $w''(s, x) = q(s, x)w(s, x)$.

Lemma 5.1 and Lemma 5.2 of [3] remain true in our current situation, that is, we have

Lemma 3.1. *There exists a constant $M > 0$ such that $\|w(s, \cdot)\|_{L^\infty(0,1)} \leq Ms^{1/2}$, $\forall s \geq 1$.*

Lemma 3.2. *Let k, r be positive constants and W be a C^2 function satisfying*

$$\Delta W(x) = Q(x)W(x) > 0, \quad Q(x) \geq 2k^2, \quad \forall x \in (-r, r).$$

Then, $W(0) \leq e^{1-kr} \max\{W(-r), W(r)\}$.

By slightly modifying the argument of Lemma 5.3 of [3], one can use Lemma 3.2 to deduce

Lemma 3.3. *Let a and R be as in (3.1). Then for any $s \geq 1$, there holds*

$$(3.4) \quad w(s, x) \leq e^{1-s^{\frac{1}{2}}(|x-x_0|-(\frac{3}{as})^{\frac{1}{2k^*-2}})} \max_{B(x_0, 2|x-x_0|)} w(s, y), \quad \forall x \in B(x_0, 2R).$$

With the help of the above lemmas, we are now ready to give

Proof of Theorem 1.4 (1-i) Clearly, under our assumption on m at $x = x_0$, one can find positive constants a and R such that (3.1) holds. Thus, w satisfies (3.4).

For any fixed $s \geq 1$, we denote

$$W(x_0, s; y) = s^{-\frac{1}{2k^*}} w(s, x_0 + s^{-\frac{1}{k^*}} y),$$

and

$$Q(x_0, s; y) = s^{-\frac{2}{k^*}} \left\{ s^2 \left| m'(x_0 + s^{-\frac{1}{k^*}} y) \right|^2 + sm''(x_0 + s^{-\frac{1}{k^*}} y) + c(x_0 + s^{-\frac{1}{k^*}} y) - \lambda_1^{\mathcal{N}}(s) \right\}.$$

Then, due to (3.1), (3.2) and (3.3), simple computation gives

$$\Delta_y W = Q(x_0, s; y)W, \quad \forall y \in B(0, s^{\frac{1}{k^*}} R),$$

and

$$ay^{2(k^*-1)} - Ay^{k^*-2} - 1 \leq Q(x_0, s; y) \leq M_2^2 y^{2(k^*-1)} + M_2 y^{k^*-2} + 1, \quad \forall y \in B(0, s^{\frac{1}{k^*}} R),$$

with $M_2 = \max_{[0,1]} |m^{(k^*)}(x)|$. In addition, combining Lemma 3.1 and Lemma 3.3, we have

$$(3.5) \quad \begin{aligned} W(x_0, s; y) &= s^{\frac{1}{2k^*}} w(s, x_0 + s^{-\frac{1}{k^*}} y) \\ &\leq M s^{\frac{k^*-1}{2k^*}} e^{1-s^{\frac{k^*-2}{2k^*}} |y| + (\frac{3}{a})^{\frac{1}{2(k^*-1)}} s^{\frac{k^*-2}{2k^*}}}, \quad \forall y \in B(0, 2s^{\frac{1}{k^*}} R). \end{aligned}$$

Let y_0 be the maximum point of $W(x_0, s; y)$ on $\overline{B}(0, 2s^{\frac{1}{k^*}} R)$, that is, y_0 satisfies

$$W(x_0, s; y_0) = \overline{M}(x_0, s, R) = \max_{y \in \overline{B}(0, 2s^{\frac{1}{k^*}} R)} W(x_0, s; y).$$

From Lemma 3.3, it then follows that

$$(3.6) \quad W(x_0, s; y) \leq \overline{M}(x_0, s, R) e^{1-s^{\frac{k^*-2}{2k^*}} |y| + (\frac{3}{a})^{\frac{1}{2(k^*-1)}} s^{\frac{k^*-2}{2k^*}}}, \quad \forall y \in B(0, s^{\frac{1}{k^*}} R).$$

We are going to show that $\overline{M}(x_0, s, R)$ is bounded, independent of all large s . For such purpose, we need consider two different cases as follows.

Case 1. $y_0 \in \partial B(0, 2s^{\frac{1}{k^*}} R)$. Thanks to (3.5) and the fact of $\frac{1}{2} > \frac{k^*-2}{2(k^*-1)}$, one yields

$$\overline{M}(x_0, s, R) \leq M s^{\frac{k^*-1}{2k^*}} e^{1-2Rs^{\frac{1}{2}} + (\frac{3}{a})^{\frac{1}{2(k^*-1)}} s^{\frac{k^*-2}{2k^*}}} \leq M_0$$

for some positive constant M_0 , which is independent of all large s and may vary from place to place below.

Case 2. $y_0 \in B(0, 2s^{\frac{1}{k^*}} R)$. Clearly $\Delta_y W(x_0, s; y_0) \leq 0$, and so $Q(x_0, s; y_0) \leq 0$. Thus,

$$a y_0^{2(k^*-1)} - A y_0^{k^*-2} - 1 \leq 0.$$

This implies $|y_0| \leq y^*$ for some constant $y^* > 0$, independent of all large s . Furthermore, for all sufficiently large s and any $y \in B(0, y^*)$, it is easy to check that the dominant term of Q is given by

$$(3.7) \quad \begin{aligned} Q(x_0, s; y) &\approx s^{-\frac{2}{k^*}} \left\{ s^2 \left| m'(x_0 + s^{-\frac{1}{k^*}} y) \right|^2 + s m''(x_0 + s^{-\frac{1}{k^*}} y) \right\} \\ &\approx s^{-\frac{2}{k^*}} \cdot s^2 (m^{(k^*)}(x_0))^2 \cdot y^{2(k^*-1)} s^{-\frac{2(k^*-1)}{k^*}} + s^{-\frac{2}{k^*}} \cdot s \cdot m^{(k^*)}(x_0) \cdot y^{k^*-2} s^{-\frac{k^*-2}{k^*}} \\ &= (m^{(k^*)}(x_0))^2 y^{2(k^*-1)} + m^{(k^*)}(x_0) y^{k^*-2}. \end{aligned}$$

This shows that $Q(x_0, s; y)$ is bounded, uniformly in all large s .

On the other hand, a direct application of the elliptic Harnack inequality (see, for instance, [9]) to the equation satisfied by W concludes that

$$\max_{y \in \overline{B}(0, y^*)} W(x_0, s; y) \leq C_0 \min_{y \in \overline{B}(0, y^*)} W(x_0, s; y)$$

for some positive constant $C_0 = C_0(a, A, M_2)$. As a result, for all large s , we get

$$\frac{\overline{M}^2(x_0, s, R)}{C_0^2} \leq \int_{B(0, y^*)} W^2 dy \leq \int_{B(0, s^{\frac{1}{k^*}} R)} W^2 dy = \int_{B(x_0, R)} w^2 dy \leq 1,$$

which implies that $\overline{M}(x_0, s, R)$ is bounded, uniformly in all large s .

In summary, the above analysis shows that $\overline{M}(x_0, s, R)$ is bounded, uniformly in all large s . With loss of generality, we now assume that $\mu(B(x_0, R)) > 0$. Then, there is a sequence $\{s_j\}$ and a positive function W^* such that

$$\lim_{j \rightarrow \infty} W(x_0, s_j; y) = W^*(y) \text{ locally uniformly in } \mathbb{R},$$

and

$$\begin{aligned} \int_{-\infty}^{\infty} (W^*)^2 dy &= \lim_{j \rightarrow \infty} \int_{B(0, (s_j)^{\frac{1}{k^*}} R)} W^2(x_0, s_j; y) dy = \lim_{j \rightarrow \infty} \int_{B(x_0, R)} w^2(s_j, x) dx \\ &= \mu(B(x_0, R)) = \mu(\{x_0\}) > 0. \end{aligned}$$

In addition, from the equation satisfied by W , W^* solves the following ODE equation

$$(W^*)''(y) = ((m^{(k^*)}(x_0))^2 y^{2(k^*-1)} + m^{(k^*)}(x_0) y^{k^*-2}) W^*(y), \quad y \in \mathbb{R}.$$

This ends the proof of Theorem 1.4(1-i). \square

Proof of Theorem 1.4 (1-ii) Assume that $x_0 = 0$ or $x_0 = 1$. The proof is similar to the case of $x_0 \in (0, 1)$, and so we only sketch it. Denote

$$W(x_0, s; y) = (\mu(s, R))^{-\frac{1}{2}} s^{-\frac{1}{2k^*}} w(s, x_0 + s^{-\frac{1}{k^*}} y),$$

and

$$Q(x_0, s; y) = s^{-\frac{2}{k^*}} \left\{ s^2 \left| m'(x_0 + s^{-\frac{1}{k^*}} y) \right|^2 + s m''(x_0 + s^{-\frac{1}{k^*}} y) + c(x_0 + s^{-\frac{1}{k^*}} y) - \lambda_1^{\mathcal{N}}(s) \right\},$$

where

$$\mu(s, R) = \int_{B(x_0, R) \cap (0, 1)} w^2(s, x) dx,$$

and $R > 0$ is chosen such that $\mu(B(x_0, R) \cap (0, 1)) = \mu(\{x_0\}) > 0$. Note that $\lim_{s \rightarrow \infty} \mu(s, R) = \mu(\{x_0\})$. As in the case of $x_0 \in (0, 1)$, we have

$$\Delta_y W = Q(x_0, s; y) W, \quad \forall y \in B(0, s^{\frac{1}{k^*}} R) \cap \Omega_s,$$

and

$$a y^{2(k^*-1)} - A y^{k^*-2} - 1 \leq Q(x_0, s; y) \leq M_2^2 y^{2(k^*-1)} + M_2 y^{k^*-2} + 1, \quad \forall y \in B(0, s^{\frac{1}{k^*}} R) \cap \Omega_s,$$

for some positive constants a and A , where $\Omega_s = \{y : x_0 + s^{\frac{1}{k^*}} R \in (0, 1)\}$ and $M_2 = \max_{[0, 1]} |m^{(k^*)}(x)|$.

Thus, the argument of Theorem 2 (ii) of [3] can be adapted to conclude that there is a sequence $\{s_j\}$ and a positive function W_* such that

$$\lim_{j \rightarrow \infty} W(x_0, s_j; y) = W_*(y) \text{ locally uniformly in } \mathbb{R}_*,$$

where $\mathbb{R}_* = (0, \infty)$ if $x_0 = 0$ and $\mathbb{R}_* = (-\infty, 0)$ if $x_0 = 1$, and W_* satisfying $\int_{\mathbb{R}_*} (W_*)^2 dy = 1$ solves the following linear ODE problem

$$(W_*)''(y) = ((m^{(k^*)}(x_0))^2 y^{2(k^*-1)} + m^{(k^*)}(x_0) y^{k^*-2}) W_*(y), \quad y \in \mathbb{R}_*.$$

\square

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